University of Thi-Qar Journal Of Science (UTsci)
Website: http://jsci.utq.edu.iq **Email: utjsci@utq.edu.iq**
Volume 6, Number 2, June 2017

Characterization of Covering Dimension

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Abstract

The present study focus and define a new kind of covering dimension and show some relations with other concepts using the ($N-open$) sets in topological space. The current paper obtain some properties and characterization of this covering dimension.

Keywords : $N-open$, *normal space, covering dimesion* .

1-Introduction:

The dimension theory begin with "dimension function" which is a role d defined on the class of topological spaces such as $d(X)$ is an integer or ∞ ,with the properties that $d(X)=d(Y)$ if X and Y are homeomorphic and $d(R^n)=n$ for each positive integer n. The dimension functions take topological spaces to the set {-1,0,1,…}.The dimension functions ind, Ind, dim investigation by A.R. Pears 1975 [2].

We define a new type of covering dimension and clear some of relations to other concepts using the N-open sets in topological space and recall the definitions of (dim). Then we introduce the dimension functions, $N-dim$ using $N-open$ sets. Follows by studing some relation between them. some results relating these concepts are proved .

2- $N-OPEN$ $SETS$

Al Omari A. and Noorani M. in [1] introduce new class of set called $N-open$ sets.

Prove that the family of all $N-open$establishes a topology. Moreover, they obtain a characterization and preserving theorems of compact spaces.

Definition 2.1[1]: A subset A of a space X is said to be $N-$open if for every $x\in A$,there exists an open subset $U_x \subseteq X$ containing x such that $U_x - A$ is a finite set. The complement of a $N-open$ subset is said to be $N-closed$ *and denoted by* $\overline{A}^N$.

The family of all $N-open$ subsets of a space (X,τ) is denoted by τ^N .

Clearly every open is $N-open$ but the converse is not true, see the following example.

Example2.2.: Let $X=\{a,b,c\}, \tau=\{\{a\}, X, \phi\}$.The $N-open$sets are:

$\phi, X, \{a\}, \{b\}, \{a,c\}, \{a,b\}, \{c\}, \{b,c\}$. Then $\{a,b\}$ is an $N-open$ set , but it is not an open.

Theorem 2.3[1]: Let (X,τ) be a topological space, then (X,τ^N) is a topological space.

Corollary2.4[1]: Let (X,τ) be a topological space. Then The intersection of an open set and $N-open$ set is $N-open$.

Proposition2.5[3]: Let X be a space ,$Y\subseteq X$ if B is an $N-open$ set in X , then $B\bigcap Y$ is an $N-open$ in Y .

Proposition2.6[4]: Let X be a space ,Y be an $N-open$ set of X **,** if A is an $N-open$ set in Y , then A is an $N-open$ in X .

Definition2.7: A space X is called $N-normal$ space if for every disjoint closed sets F_1, F_2 there exist disjoint $N-open$sets V_1, V_2 such that $F_1 \subseteq V_1, F_1 \subseteq V_1$.

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Definition2.8: A space X is called $N^*-normal$ space if for every disjoint $N-closed$ sets F_1, F_2 there exist disjoint open sets V_1, V_2 such that $F_1 \subseteq V_1$, $F_2 \subseteq V_2$.

Remark2.9: It is clear that $N^*-normal$ space is normal, and every normal space is $N-normal$ space .

Example 2.10: This example shows that a $N-normal$ space is not need to be normal in . Let $X = \{a,b,c,d,e\}$, $\tau = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{a,b,d,e\}, \{d\}, \{a,d\}\}$. The $N-open$ sets are (every sub sets of X) . It is clear that X is $N-normal$ space but is not normal. In Fact the closed sets $\{c\}, \{b,e\}$ cannot be separated by open sets in X .

Remark 2.11: Example **2.2** shows that a normal space is not $N^*-normal$ in general . It is clear that X is normal space since there exist no disjoint closed sets . Hence it is $N-normal$ since every normal space is $N-normal$ on the other hand , X is not $N^*-normal$ since there are no disjoint open sets which separate the $N-closed$ sets $\{b\}, \{c\}$.

Proposition 2.12: A space X is $N-normal$ space if for every closed set F in X and open set U such that $F \subseteq X$ there exists an $N-open$ set W such that $F \subseteq W \subseteq \overline{W}^N \subseteq U$.

Proof: It is clear that F, U^c are disjoint closed set in X . Thus since X is $N-normal$ space then there exist disjoint $N-open$ sets W, V such that $F \subseteq W$, $U^c \subseteq V$ then $F \subseteq W \subseteq \overline{W}^N \subseteq \overline{V^c}^N = V^c \subseteq U$. Conversely, let F_1, F_2 be disjoint closed sets in X . Then F_2^c is open set , $F_1 \subseteq F_2^c$. Thus there exists an $N-open$ set W such that $F_1 \subseteq W \subseteq \overline{W}^N \subseteq F_2^c$. Then $F_1 \subseteq W, F_2 \subseteq \overline{W}^{N^c}$, $W, \overline{W}^N$ are disjoint $N-open$ sets . So that X is $N-normal$ space .

By the same technique we can prove the following Proposition .

Proposition2.13: A space X is $N^*-normal$ space if and only if for every $N-closed$ set F in X and $N-open$ set U such that $F \subseteq U$, there exists an open set W such that $F \subseteq W \subseteq \overline{W}^N \subseteq U$.

Theorem 2.14: Let X be a topological space, Then the following statements are equivalent:

(a) X is $N^*-normal$.

(b) Each point – finite $N-open$ covering of X is shrinkable .

(c) Each finite $N-open$ covering of X has locally finite closed refinement.

Proof:(a) →(b) Let $\{G_\lambda\}_{\lambda \in \wedge}$ be a point – finite $N-open$ covering of $N^*-normal$ space X and let $\wedge$ be well – ordered . We shall construct a shrinkable of $\{G_\lambda\}_{\lambda \in \wedge}$ by transfinite induction . Let μ be an element of Λ and suppose that for each $\lambda < \mu$. We have an open set U_λ such that $\overline{U}_\lambda \subset G_\lambda$ for each $\lambda < \mu$, $\bigcup_{\lambda \le \mu} U_\lambda \cup \bigcup_{\lambda > \mu} G_\lambda = X$. Let x be a point in X . Then since $\{G_\lambda\}_{\lambda \in \wedge}$ is point finite , there exists a largest $\sum$, say , of $\wedge$ such that $x \in G_\Sigma$. If $\sum \ge \mu$ then $x \in \bigcup_{\lambda \ge \mu} G_\lambda$, whilst if $\sum < \mu$ then $x \in \bigcup_{\lambda \le \Sigma} U_\lambda \subset \bigcup_{\lambda < \mu} U_\lambda$. Hence $\bigcup_{\lambda < \mu} U_\lambda \cup \bigcup_{\lambda \ge \mu} G_\lambda = X$. Thus G_μ contains the complement of $\bigcup_{\lambda < \mu} U_\lambda \cup \bigcup_{\lambda > \mu} G_\lambda$ since X is $N^*-normal$,there exists an open set U_μ such that $X \setminus (\bigcup_{\lambda < \mu} U_\lambda \cup \bigcup_{\lambda > \mu} G_\lambda) \subset U_\mu \subset \overline{U}_\mu \subset G_\mu$. Thus $\overline{U}_\mu \subset G_\mu$ and $\bigcup_{\lambda \le \mu} U_\lambda \cup \bigcup_{\lambda > \mu} G_\lambda = X$. The construction of a shrinking of $\{G_\lambda\}_{\lambda \in \wedge}$ is completed by transfinite induction .

(b) →(c) Let $\{G_\alpha : \alpha \in \wedge\}$ be a finite $N-open$ covering of X , then $\{G_\alpha : \alpha \in \wedge\}$ is a point – finite open covering of X therefore, there exists $\{U_\alpha : \alpha \in \wedge\}$ an open family of covering of X , such that $\overline{U}_\alpha \subset G_\alpha$ for each $\alpha \in \wedge$. Therefore $\{\overline{U}_\alpha : \alpha \in \wedge\}$ is a locally finite closed refinement of $\{G_\alpha : \alpha \in \wedge\}$.

(c) →(a) Let X be a space such that each finite $N-open$ covering of X which has a locally finite closed refinement and let A, B be disjoint $N-closed$

sets of X. The covering $\{X \setminus A, X \setminus B\}$ of X has a locally finite closed refinement F. Let E be the union of the members of F disjoint from A and let G be the union of the members of F disjoint from B, then E and G are closed sets and $E \bigcup G = X$. Thus if $U = X \setminus E, W = X \setminus C$, then U, W are disjoint open sets $A \subseteq U$, $B \subseteq W$. Hence X is $N^* - normal$space .

3- *On N - Covering Dimension* (*N-dim*):

Definition 3.1[2]: The order of a family $\{A_\lambda\}_{\lambda\in\wedge}$ of subsets , not all empty , of some set is the largest integer n for which exists a subset μ of $\wedge$ with $n+1$ elements such that $\bigcap_{\lambda\in\mu} A_\lambda$ is not empty , or is ∞ if there is no such largest integer. A family of empty subset has order -1.

Definition 3.2 [2]**:** Let X be a topological space , then $dim\ X = -1$ if and only if $X = \phi$, and if n is a positive integer or 0 then we say that $dim\ X \le n$ if and only if every finite open cover of X has an open refinement of order $\le n$ or is ∞ if there is no such integer .

This suggests the following definition:

Definition 3.3: Let X be a topological space , then $N - dim\ X = -1$ if and only if $X = \phi$, and if n is a positive integer or 0 then we say that $N - dim\ X \le n$ if and only if every finite open cover of X has an $N - open$ refinement of order $\le n$ or is ∞ if there is no such integer .

Remark 3.4: Since each open set is $N - open$, then it follows that $N - dim\ X \le dim\ X$.

Theorem 3.5: Let X be a topological space. If X has a base of sets which are both $N - open$ and $N - closed$, then $N - dim\ X = 0$. For a $T_1 - space$ the converse is true .

Proof: Suppose X has a base of sets which are both $N - open$ and $N - closed$. Let $\{U_i\}_{i=1}^k$ be a finite open covering of X. It has an $N - open$ refinement W, if $W \in$ W then $W \subset U_i$ for some i. Let each W in W be associated with one of the sets U_i containing it and let V_i be the union of those members of W thus associated with U_i. Thus Vi is $N - open$ set , and hence $\{Vi\}_{i=1}^k$ forms a disjoint $N - open$ refinement of $\{U_i\}_{i=1}^k$. Then $N - dim\ X = 0$. Conversely suppose X is a $T_1 - space$ such that $N - dim\ X = 0$. Let $x \in X$ and G be an open set in X such that $x \in G$. Then $\{x\}$ is closed and $\{G, X - \{x\}\}$ is a finite of open cover of X. So it has an $N - open$ refinement of order 0. Let C_1 be the union of $N - open$ sets in G and C_2 be the union of the $N - open$ sets in $X - \{x\}$. Then $C_1 \bigcap C_2 = \phi$, $C_1 \bigcup C_2 = X$ and C_1, C_2 are $N - open$, and $N - closed$ set in X. Thus $x \in C_2^c = C_1 \subseteq G$ and C_1 is $N - open$ and $N - closed$ set in X and hence X has a base of sets which are both $N - open$ and $N - closed$ sets.

It is known that if X is a topological space with $dim\ X = 0$ then X is normal .

Theorem3.6: Let X be a topological space. If $N - dim\ X = 0$,then X is a $N - normal$.

Proof : Let C_1, C_2 be disjoint closed sets in X , then $\{X \setminus C_1, X \setminus C_2\}$ is a finite open covering of X. Since $N - dim\ X = 0$ then it has $N - open$ refinement of order 0 say C. Let H be the union of it such that $N - open$ disjoint from C_1 and let G be the union of it such that $N - open$ disjoint from C_2. Then H, G are $N - open$ sets , $H \bigcup G = X$, $H \bigcap G = \phi$ so that $H \subseteq X \setminus C_1, G \subseteq X \setminus C_2$. Thus $C_1 \subseteq H^c = G$ and $C_2 \subseteq G^c = H$ and since $H \bigcap G = \phi$, then X is $N - normal$ space .

Remark 3.7: Let $X = \{a,b,c\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a,c\}\}$. In this example show that $dim\ X = N - dim\ X = 0$. Since X is the open cover of X and it is the only open refinement of it , then $dim\ X = 0$ and since $N - dim\ X \le dim\ X, X \ne \phi$, then $dim\ X = N - dim\ X = 0$.

The following example shows that $dim\ X = N - dim\ X = 1$.

Example 3.8: Let $X = \{a,b,c,d\}$ and let a base for a topology of X consisting of the sets $\{a\}$, $\{d\}$, $\{b,d\}$ and $\{c,d\}$.Then $\{\{a\},\{b,d\},\{c,d\}\}$ is an open and $N - open$ refinement for every open covering of X. So that $dim\ X \le 1$ and $N - dim\ X \le 1$. But X is non empty , not normal and not $N - normal$[since $\{a,c\}, \{b\}$ are disjoint closed sets but there is no disjoint open or $N-open$ sets G,H such that $\{a,c\}\subseteq H$, $\{b\}\subseteq G$] so that $dim\ X>0$, $N-dim\ X>0$. Hence $dim\ X=N-dim\ X=1$.

The following example shows that $dim\ X\neq N-dim\ X$ in general:

Example3.9: Suppose $X_m=\{(x,y)\in R^2 : y=mx\ , m\in Z^+, y>0, x>0\}$, and let $X=\{0\}\ \bigcup(\bigcup_{m=1}^{k} X_m)$. Let a_m be the point of intersection of the line $y=mx$ with the circumference of the unit open disc D with center 0, $a_m\notin D$. Denote the topology of X_m by T_m, take a base for a point $x\in X_m$, $x\neq a_m$ to be the family of open intervals containing x but not a_m, and the base for a_m is X_m . Let T be the topology on X generated by $\bigcup_{m=1}^{k} T_m$ and the base at 0 family D. It is clear that X is not normal space , since $\{0\}, D^c$ are disjoint closed sets but there exist no disjoint open sets separate them . the finite open cover of X are X or $\{X_m : m=1,.....,k\}\bigcup D$, and hence $\{X_m : m=1,.....,k\}\bigcup D$ is a finite open refinement for every open cover of X which is of order ≤ 1 and since X is not normal , then $dim\ X>0$ and hence $dim\ X=1$.Now let $\{G_\lambda\}$ be a finite an open cover of X , if one member $G_\lambda=X$ then $\{X\}$ is a finite refinement of $N-open$ sets and $N-dim\ X=0$, otherwise at last one $G_\lambda\ni D$ call it $G_{\lambda\circ}\ni D$. Moreover for each m, at least one G_λ say $G_{\lambda m}\ni X_m$ because the only open set containing a_m is X_m. There is no loss of generality if we suppose that G_λ is an open interval when $\lambda\neq\lambda\circ, \lambda_1...\lambda_m,...,\lambda_k$ each $X_m\setminus\{a_m\}$ is a collection of open intervals: $D\bigcup\{[a_1,\infty),...,[a_k,\infty)\}$ is an $N-open$ refinement, since each $[a_i,\infty)$ is $N-open$ set for each i and since this $N-open$ refinement are disjoint, then $N-dim\ X=0$. Thus $dim\ X\neq N-dim\ X$.

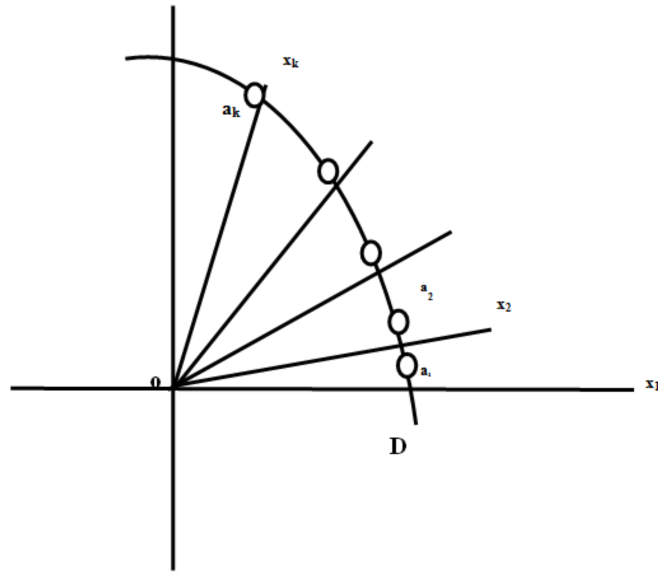

Figure Shows $dim\ X\neq N-dim\ X$

Theorem 3.10: If A is both open and closed subset of X then $N-dim\ A\leq N-dim\ X$.

Proof : Suppose that $N-dim\ X\leq n$. Let $\{U_1,......,U_k\}$ be an open covering of A . Then for each i ,$U_i=A\bigcap V_i$ where V_i is an open set in X . The finite open covering $\{V_1,...,V_k, X\setminus A\}$ of X has an $N-open$ refinement W of order $\leq n$. Let $V=\{W\bigcap A\setminus W\in W\}$. Then V is an $N-open$ refinement of $\{U_1,......,U_k\}$ of order $\leq n$. Thus $N-dim\ A\leq n$.

Theorem 3.11[2]: If X is *normal* space, the following statements are equivalents :

(a) $dim\ X\leq n$

(b) For each family of closed sets $\{C_1,...,C_{n+1}\}$ and each family of open set $\{U_1,...,U_{n+1}\}$ such that $C_i\subset U_i$ there exists a family $\{V_1,...,V_{n+1}\}$ of open sets such that $C_i\subset V_i\subset\overline{V_i}\subset U_i$ for each i and $\bigcap_{i=1}^{n+1} b(V_i)=\phi$.

(c) for each family of closed sets $\{C_1,...,C_k\}$and each open family of open sets $\{U_1,...,U_k\}$ such that each $C_i\subset U_i$ there exists families $\{V_1,...,V_k\}$ and $\{W_1,...,W_k\}$ of open sets such that

$C_i \subset V_i \subset \overline{V}_i \subset W_i \subset U_i$ for each i and the order of the family $\{\overline{W}_1/V_1,...,\overline{W}_k/V_k\}$ does not exceed $n-1$.

Theorem 3.12: If X is $N^*-normal$ space , the following statements are equivalents :

(a) $N-dim\ X \le n$

(b) For each family of closed sets $\{C_1,...,C_{n+1}\}$ and each family of open set $\{U_1,...,U_{n+1}\}$ such that $C_i \subset U_i$ there exists a family $\{V_1,...,V_{n+1}\}$ of open sets such that $C_i \subset V_i \subset \overline{V}_i \subset U_i$ for each i and $\bigcap_{i=1}^{n+1} b(V_i) = \phi$.

(c) for each family of closed sets $\{C_1,...,C_k\}$and each open family of open sets $\{U_1,...,U_k\}$ such that each $C_i \subset U_i$ there exists families $\{V_1,...,V_k\}$ and $\{W_1,...,W_k\}$ of open sets such that $C_i \subset V_i \subset \overline{V}_i \subset W_i \subset U_i$ for each i and the order of the family $\{\overline{W}_1/V_1,...,\overline{W}_k/V_k\}$ does not exceed $n-1$.

Proof:(a)→(b) Suppose that $N-dim\ X \le n$. Let $C_1,...,C_{n+1}$ be closed sets and let $U_1,...,U_{n+1}$ be open sets such that each $C_i \subset U_i$. Since $N-dim\ X \le n$, the open covering of X consisting of sets of the form $\{H_1,...,H_{n+1}\}$, where $H_i = U_i$ or $H_i = X \setminus C_i$ for each i , has a finite $N-open$ refinement $\{W_1,...,W_q\}$ of order not exceeding n . Since X is $N^*-normal$, there is a closed covering $\{K_1,...,K_q\}$ such that each $K_r \subset W_r$ for each $r=1,...,q$. Let N_r denote the set of integers i such that $C_i \bigcap W_r \ne \phi$ for $r=1,...,q$, we can find open sets V_{ir} for i in N_r such that $K_r \subset V_{ir} \subset \overline{V}_{ir} \subset W_r$ and $\overline{V}_{ir} \subset V_{jr}$ if $i<j$. Now for each $i=1,...,n+1$, let $V_i = \bigcup_r \{V_{ir} \setminus i \in N_r\}$. Then V_i is open , and $C_i \subset V_i$, for if $x \in C_i$ and $x \in K_r$, then $i \in N_r$ so that , $x \in V_{ir} \subset V_i$. Furthermore if $i \in N_r$ so that $C_i \bigcap W_r \ne \phi$, then W_r is not contained in $X \setminus C_i$ so that $W_r \subset U_i$. Thus if $i \in N_r$, then $V_{ir} \subset U_i$ so that , since $\overline{V}_i = \bigcup_r \{\overline{V}_{ir} \setminus i \in N_r\}$, it follows that $\overline{V}_i \subset U_i$. Finally suppose that $x \in \bigcap_{i=1}^{n+1} b(V_i)$. Since $b(V_i) \subset \bigcup_r \{b(V_{ir}) \setminus i \in N_r\}$, it follows that for each i there exists r_i such that $x \in b(V_{ir_i})$. And if $i \ne j$, then $r_i \ne r_j$ for if $r_i = r_j = r$ then $x \in \overline{V}_{ir}$ and $x \in \overline{V}_{jr}$ but $x \notin \overline{V}_{ir}$ and $x \notin \overline{V}_{jr}$, which is contradiction , since either or $\overline{V}_{jr} \subset V_{ir}$. For each i , $x \notin V_{ir_i}$ so that $x \notin K_{ri}$. But $\{K_r\}$ is a covering of X and so there exists $r_\circ$ different from each of the r_i such that $x \in K_{r_\circ} \subset W_{r_\circ}$. Since $x \in \overline{V}_{ir_i}$, it follows that $x \in W_{ri}$ for $i=1,...,n+1$, so that $x \in \bigcap_{i=0}^{n+1} W_{ri}$. Since the order of $\{W_r\}$ does not exceed n. Hence $\bigcap_{i=1}^{n+1} b(V_i) = \phi$.

(b)→(c) Since each $N^*-normal$ space is normal , then (b)→(c) by Proposition **3.12** .

(c)→(a) Since each $N^*-normal$ space is normal . then we get $N-dim\ X \le n$ by Proposition **3.12**. And since $N-dim\ X \le dim\ X$, hence $N-dim\ X \le n$.

Theorem (Uryshon's Lemma) 3.13[5]:For every pair A,B of disjoint closed subsets of normal space X there exist a continuous function $f: X \to I$ such that $f(x)=0$ for $x \in A$ and $f(x)=1$ for $x \in B$, where $I=[0,1]$.

Proposition 3.14[2]:If X **is** $normal$ space, the following statements about X are quivalents:

(a) $dim\ X \le n$.

(b) for each family of $n+1$ pairs of closed sets $\{(E_1,F_1),...,(E_{n+1},F_{n+1})\}$ where $E_i \bigcap F_i = \phi$ for each i, there exist $n+1$ continuous function $f_i : X \to [-1,1], i=1,...n+1$ such that for each i , $f_i(x)=1$ if $x \in E_i$, $f_i(x)=-1$ if $x \in F_i$ and $\bigcap_{i=1}^{n+1} f_i^{-1}(0) = \phi$.

(c) for each family of $n+1$ pairs of closed sets $\{(E_1,F_1),...,(E_{n+1},F_{n+1})\}$ where $E_i \bigcap F_i = \phi$ for each i, there exists a family $\{C_1,...,C_{n+1}\}$ of closed sets such that each C_i separated E_i and F_i in and $\bigcap_{i=1}^{n+1} C_i = \phi$.

Theorem 3.15: If X is $N^*-normal$ space, the following statements about X are equivalents:

(a) $N-dim\ X \le n$.

(b) for each family of $n+1$ pairs of closed sets $\{(E_1,F_1),...,(E_{n+1},F_{n+1})\}$ where $E_i \bigcap F_i = \phi$ for each i, there exist $n+1$ continuous function $f_i : X \to [-1,1], i=1,...n+1$ such that for each i, $f_i(x)=1$ if $x \in E_i$, $f_i(x)=-1$ if $x \in F_i$ and $\bigcap_{i=1}^{n+1} f_i^{-1}(0)=\phi$.

(c) for each family of $n+1$ pairs of closed sets $\{ (E_1,F_1),...,(E_{n+1},F_{n+1}) \}$ where $E_i \cap F_i = \phi$ for each i, there exists a family $\{C_1,...,C_{n+1}\}$ of closed sets such that each C_i separated E_i and F_i in and $\bigcap_{i=1}^{n+1} C_i = \phi$.

Proof :(a)→(b) If X is $N^* - normal$ space such that $N - dim\ X \le n$, and let $\{(E_1,F_1),...,(E_{n+1},F_{n+1})\}$ be a family of pairs of disjoint closed sets. By theorem 2.13 there exist open sets $V_1,...,V_{n+1}$ and $W_1,...,W_{n+1}$ such that $E_i \subset V_i \subset \overline{V}_i \subset W_i \subset X \setminus F_i$ and $\bigcap_{i=1}^{n+1}(\overline{W}_i / V_i) = \phi$ .By Urysohn's Lemma, for each i there exists a continuous function $f_i : X \to [-1,1]$ such that $f_i(x)=1$ if $x \in \overline{V}_i$ and $f_i(x)=-1$ if $x \notin W_i$. We note that $f_i^{-1}(0) \subset W_i \setminus \overline{V}_i \subset \overline{W}_i \setminus V_i$.. Thus we have $n+1$ continuous functions $f_i : X \to [-1,1]$ such that $f_i(x)=1$ if $x \in E_i$, $f_i(x)=-1$ if $x \in F_i$ and $\bigcap_{i=1}^{n+1} f_i^{-1}(0)=\phi$.

(b)→(c) Since each $N^* - normal$ is normal , then (b)→(c)by Proposition **3.14** .

(c)→(a) Since each $N^* - normal$ space is normal, then we get $dim\ X \le n$ by Proposition **3.14 .** And since $N - dim\ X \le dim\ X$. Hence $N - dim\ X \le n$.

Lemma 3.16[4]: If X is a normal ,let A be a closed sub space of X and let the continuous function $f_0, f_1 : A \to S^n$ be uniformly homotopic. If f_0 has an extension $g_0 : X \to S^n$, then f_1 has an extension $g_1 : X \to S^n$.

Theorem 3.17[4]: If X is a normal , then $dim\ X \le n$ iff for each closed set A of X, each continuous function $f : A \to S^n$ has an extension $g : X \to S^n$.

Theorem3.18: If X is $N^* - normal$, then $N - dim\ X \le n$ iff for each closed set A of X, each continuous function $f : A \to S^n$ has an extension $g : X \to S^n$.

Proof : Let X be a $N^* - normal$ space such that $N - dim\ X \le n$, let A be a closed subspace of X and let $f : A \to S^n$ be given continuous function. We regard S^n as the boundary of the cube Q^{n+1} in R^{n+1} ,where $Q^{n+1} = \{t \in R^{n+1} \setminus \|t\| \le 1\ for\ i=1, ...,n+1\}$.If $x \in A$ let $f(x) = (f_1(x),..., f_{n+1}(x))$ and for $i=1,....,n+1$ let $E_i = \{x \in A \setminus f_i(x)=1\}$, $F_i = \{x \in A \setminus f_i(x)=-1\}$. Then E_i, F_i are disjoint sets , closed in A and hence in X , and $A = \bigcup_{i=1}^{n+1}(E_i \cup F_i)$ .By Theorem **3.15** there exist continuous function $\eta_i : X \to [-1,1]$, $i=1,...,n+1$, such that $\eta_i(x)=1$ if $x \in E_i$, $n_i(x)=-1$ if $x \in F_i$ and $\bigcap_{i=1}^{n+1} \eta_i^{-1}(0)=\phi$. Let $\eta : X \to Q^{n+1}$ by given by $\eta(x) = (\eta_1(x),....,\eta_{n+1}(x))$. If $x \in A$, then either $x \in E_i$ for some i so that $\eta_i(x) = f_i(x) = 1$ or , $x \in F_j$ for some j, $\eta_j(x) = f_j(x) = -1$. It follows that we can define continuous functions $\Psi : A \to S^n$ and $h : A \times I \to S^n$ by putting $\Psi(x) = \eta(x)$ if $x \in A$ and $h(x,t) = (1-t)\Psi(x) + tf(x)$ if $(x,t) \in A \times I$. If $x \in A$ and $s,t \in I$, then $\|h(x,s) - h(x,t)\| = |s-t|\ \|\Psi(x) - f(x)\|$. Since $\|\Psi(x) - f(x)\| \le 2\sqrt{n-1}$ if $x \in A$, it follows that h is a uniform homotopy between Ψ and f . Since $\bigcap_{i=1}^{n+1} \eta_i^{-1}(0)=\phi$, it follows that $\eta(x) \subset Q^{n+1} \setminus \{0\}$ so that Ψ has an extension to X since S^n is a retract of $Q^{n+1} \setminus \{0\}$. It now follows from lemma **3.16** that f has an extension $g : X \to S^n$. Conversely since each $N^* - normal$space is normal. Then $dim\ X \le n$ from Theorem **3.17** since $N - dim\ X \le dim\ X$. Hence $N - dim\ X \le n$.

Proposition 3.19[2]: If X is a normal, let A be a closed sub space of X and let $f : A \to S^n$ be a continuous function . Then there exist an open set U and a continuous $g : U \to S^n$ such that $A \subset U$ and $g|\ A = f$.

Proposition 3.20[2]: Let A be a closed set of normal space X such that $dim\ C \le n$ for each closed C of X which is disjoint from A. Then each continuous function $f : A \to S^n$ has an extension $g : X \to S^n$

.Then each continuous function $f: A \to S^n$ has an extension $g: X \to S^n$.

Theorem 3.21: Let A be a closed set of a $N^* - normal$ space X such that $N - dim\ C \le n$ for each closed C of X which is disjoint from A. Then each continuous function $f: A \to S^n$ has an extension $g: X \to S^n$.

Proof : Since X is $N^* - normal$ by proposition **3.19** there exists an open set U such that $A \subset U$ and a mapping $\eta: U \to S^n$ which extends f, and there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$. The set $\overline{V} \setminus V$ is closed in $X \setminus V$ and $N - dim(X \setminus V) \le n$ since $X \setminus V$ is a closed set of X disjoint form A. Hence by Proposition **2.20** and theorem **3.18** there exists a continuous function $\Psi: X \setminus V \to S^n$ such that $\Psi|\ \overline{V} \setminus V = \eta|\ \overline{V} \setminus V$. Let $g: X \to S^n$ be define as follows :

$$g(x) = \begin{cases} \eta(x) & x \in \overline{V} \\ \Psi(x) & x \in X \setminus V \end{cases}$$

The definition is meaningful and the continuous function g is the required extension of f .

Theorem 3.22: Let X be a $N^* - normal$ space and let A be closed of X such that $N - dim\ A \le n$ if B is a closed set of X and $\eta: B \to S^n$ is continuous, then there exist an open set V such that $B \subset V$ and a continuous function $\Psi: A \bigcup \overline{V} \to S^n$ such that $\Psi|B = \eta$.

Proof: By proposition **3.19** there exists an open set U such that $B \subset U$ and a continuous function $g: U \to S^n$ such that $g|\ B = \eta$.There exist an open set V such that $B \subseteq V \subseteq \overline{V} \subseteq U$. If $\overline{V}$ does not meet A, then let $\Psi: A \bigcup \overline{V} \to S^n$ be a mapping such that $\Psi|\ A$ is a constant and $\Psi|\ \overline{A} = g|\ \overline{V}$. If $\overline{V}$ meet A , then $g|\ A \bigcap \overline{V}$ has an extension $h: A \to S^n$ since $N - dim\ A \le n$. Let $\Psi: A \bigcup \overline{V} \to S^n$ be the unique mapping such that $\Psi|\ A = h$ and $\Psi|\ \overline{V} = g|\ \overline{V}$. In both cases Ψ is the required extension .

Theorem 3.23: Let A be a closed set of $N^* - normal$ space X . If $N - dim\ A \le n$ and if $N - dim\ C \le n$ for each closed C of X which does not meet A, then $dim\ X \le n$.

Proof: Let B be a closed set of X and let $f: B \to S^n$ be a continuous function . It follows from Theorem **3.22** that f has an extension $g: A \bigcup B \to S^n$. By hypothesis if C is a closed set of X disjoint from $A \bigcup B$ then $N - dim\ C \le n$, so that by Theorem **3.21** , g has an extension $h: X \to S^n$. Then h is an extension of f . Thus $dim\ X \le n$ by Theorem **3.17**

Theorem3.24: Let X be a $N^* - normal$ space and has a countable cover $\{A_i\}_{i=1}^{\infty}$ where each A_i is closed and $N - \dim A_i \le n$ for each i , then $N - dim\ X \le n$.

Proof: Let C be a closed set of X and let $f: C \to S^n$ be a continuous function .By Theorem **3.22** those is an open V_1 such that $C \subset V_1$ and there is an extension $h_1: \overline{V_1} \cup A_1 \to S^n$ of f .Next there is an open set $V_2 \supset \overline{V_1} \cup A_1$ and there is a continuous function $h_2: \overline{V_2} \cup A_2 \to S^n$ extending h_1 (also f) .Repeating this procedure we get an extension h_i of h_{i-1} , $h_i: \overline{V_i} \cup A_i \to S^n$ .Putting $g_i = h_i|_{V_i}$ for each i, then we get a continuous function $g_1: V_1 \to S^n, g_2: V_2 \to S^n$,… such that $g_i = g_j|_{V_i}$ for every $i < j$ and g_j has an extension over X (because A_i is a cover of X) . So there is a function $g: X \to S^n$ such that $g_i = g|_{V_i}$ for each i this is continuous because each V_i is open . Hence g extends f then $N - \dim X \le n$ by Theorem **3.18** .

References:

[1] Al. Omari. A. and Noorani. S. Md. "New characterization of comoact space" proceeding of the 5th Asian Mathematical conference , Malaysia ,2009, pp.53-60.

[2] A. P. Pears, "On Dimension Theory of General Spaces" Cambridge University press ,1975

[3] Majhool, Fatima and Hamza, Sattar "On N-proper Action" M.Sc. Thesis , Al- Qadisiyah University, College of computer science and Mathematics, 2011.

[4] N. Bourbaki , Elements of Mathematics , "General Topology ", Springer Verlog, Berlin, Heidelbery, Paris New-York, London, Tokyo , 2nd Edition (1989) .

[5] R. Engelking,"General topology ", Berlin, Heldermann (1989) .