ISSN 1991- 8690 الترقيم الدولي ٨٦٩٠ - ١٩٩١
website: http://jsci.utq.edu.iq Email: utjsci@utq.edu.iq

On Quasi Prime Fuzzy Submodules and Quasi Primary Fuzzy Submodules

Rabee Hadi Jari **Hassan Kiream Hassan**
Department of mathematics - College of Education,
University of Thi-Qar.

Abstract

In this paper we give some results about fuzzy quasi prime submodules, also,we study the notion of quasi primary fuzzy submodules.

1. Introduction:

The notion of prime fuzzy submodules was introduced by Hatem Y.K [1] and some results a bout this concept are presented .In this paper we give more results about this concept and study the extension of quasi primary fuzzy submodule by fuzzifing the notion of (ordinary) quasi primary fuzzy submodule and give some characterizations about it ,also we give some properties related to it.

Through out the paper R is a commutative ring with unity and M is an R-module.

2. Preliminaries:

We record here some basic concepts and clarify notions which are used in the next work .For details,we refer to [2,3].

Definition 2.1 [2]:

A fuzzy set is a mapping X from a nonempty set M in to [0,1], then the set

$$X_t = \{x \in M \setminus X(x) \geq t\}$$

is called a level subset of X.

Definition 2.2 [3]:

Let $x_t : M \to [0,1]$ **be a fuzzy set in M, where,** $x \in M$ **,** $t \in [0,1]$, **then** x_t **is defined by :**

$$x_t(y) = \begin{cases} \text{t} & \text{if } x = y \\ \text{o} & \text{if } x \neq y \text{ for all } y \in M. \end{cases}$$

x_t **is called a fuzzy singleton.**

If x=0 and t=1, then:

Definition 2.3 [4]:

A fuzzy set X in R-module M is called fuzzy R-module if and only if each $x, y \in M$ and $r \in R$ **, then:**

$$0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

Definition 2.3 [4]:

A fuzzy set X in R-module M is called a fuzzy R-module if and only if each $x, y \in M$ and $r \in R$ **, then:**

1. $X(x-y) \geq \min\{X(x), X(y)\}$.

2. $X(rx) \geq X(x)$

3. X(0)=1. (0 is the zero element of M).

Definition 2.4 [5]:

Let A and X be two fuzzy module of an R-module M, A is called a fuzzy submodule of X if and only if $A \subseteq X$ **.**

Definition 2.5 [5]:

Let r_t and x_s be two fuzzy singletons of R and M respectively. then $r_t x_s = (rx)_\lambda$, Where λ=min{s,t}.

Definition 2.6 [5]:

Let r_t be a fuzzy singleton of R and A be a fuzzy module of R-module. Then for any $w \in M$,

$$(r_t A)(w) = \begin{cases} \sup\{\inf\{t, A(x)\}\}, & \text{if all } w = rx \text{ for some } x \in M. \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.7 [6]:

Let X be a nonempty fuzzy module of R-module M. The fuzzy annihilator of A denoted by F-annA is defined by:

$$F - annA = (0_1 : A) = \{x_t : x_t A \subseteq 0_1, x \in R, t \in [0,1]\}$$

Where A is a proper fuzzy submodule of X.

3. Quasi Prime Fuzzy submodules:

In this section we turn our attention to the extension of a quasi prime modules.

Definition 3.1 [1]:

A fuzzy submodule A of a fuzzy module X of an R-module M is called quasi prime fuzzy submodule of X if whenever $a_s b_k x_t \subseteq A$ for fuzzy singleton a_s, b_k of R and $x_t \subseteq X$, implies that $a_s b_k \subseteq A$ or $b_k x_t \subseteq A$.

Definition 3.2 [1]:

Let X be a fuzzy module of an R-module M. X is called a quasi prime fuzzy module if and only if F-annA is a prime fuzzy ideal of R for each nonempty fuzzy submodule A of X. Equivalently, F-ann(x_t)=F-ann($r_s x_t$) for each $x_t \subseteq X$ and for each fuzzy singleton r_s of R such that $r_s \not\subset F - annX$.

Definition 3.3 [6]:

Let A be a fuzzy submodule of a fuzzy module X, then A is called an essential fuzzy submodule if $A \cap B \neq 0_1$, for each nonempty fuzzy submodule B of X.

Lemma 3.4[6]:

Let A be an essential fuzzy submodule of X, if $a_k \subseteq X$, then there exists a fuzzy singleton r_t of R such that $0_1 \neq r_t a_k \subseteq A$.

we noticed that if a fuzzy module X is quasi prime fuzzy module ,then any fuzzy submodule of X is a quasi prime fuzzy module but the converse is not true,see[1,Remarks and Examples 2.1.3(4)]. However, we put certain condition under which the converse is true.

Proposition 3.5:

Let A be an essential fuzzy submodule of a fuzzy module X such that F-ann x_t=F-ann $r_s x_t$ for each $x_t \subseteq X$ and for a fuzzy singleton r_s of R such that $r_s \not\subset F - ann x_t$, then X is a quasi prime fuzzy module if and only if A is a quasi prime fuzzy module.

Proof

Suppose that A is a quasi prime fuzzy R-module.

To prove that 0_1 is a quasi prime fuzzy submodule of X.

Let $a_l b_k x_t \subseteq 0_1$

for fuzzy singletons a_l, b_k of R and $x_t \subseteq X$. Since A is an essential fuzzy submodule, thus by lemma 3.4, there exists a fuzzy singleton r_s of R such that $0_1 \neq r_s x_t \subseteq A$, that is $r_s \not\subset F - ann x_t$.

Hence $a_l b_k (r_s x_t) \subseteq 0_1$. But A is a quasi prime fuzzy R-module, thus 0_1 is a quasi prime fuzzy submodule of A by [1, proposition 3.4.1], so either

$a_l(r_s x_t) \subseteq 0_1$ or $b_k(r_s x_t) \subseteq 0_1$, thus either $a_l \subseteq F - ann r_s x_t$ or $b_k \subseteq F - ann r_s x_t$. But $F - ann r_s x_t = F - ann x_t$.

Consequently, either $a_l \subseteq F - ann x_t$ or $b_k \subseteq F - ann x_t$, that is either $a_l x_t \subseteq 0_1$ or $b_k x_t \subseteq 0_1$. Hence X is a quasi prime fuzzy module by [1, proposition 3.4.1].

The converse follows directly by [1, Remarks and Examples 2.1.3(4)].

Definition 3.6 [6]:

A fuzzy module X is called faithful if F-ann$X=0_1$.

Recall that a fuzzy ideal I of a ring R is called a prime fuzzy ideal of R if for each $r_t a_k \subseteq I$, then either $r_t \subseteq I$ or $a_k \subseteq I$, [7].

Corollary 3.7:

Let A be an essential fuzzy submodule of a fuzzy module X, if A is a faithful quasi prime module, then X is a faithful quasi prime fuzzy module.

Proof:

Since A is a faithful, then F-ann$A=0_1$, also, $A \subseteq X$, then $F - annX \subseteq F - annA$ by [6, Remark 1.2.18], which implies that $F - annX \subseteq 0_1$, but $0_1 \subseteq F - annX$, hence F-ann$X=0_1$, That is X is faithful, moreover it follows that F-annX=F-annA.

Now, to prove X is a quasi prime fuzzy module,it is enough to prove that F-ann(x_t)=F-ann($r_s x_t$) for each $x_t \subseteq X$ and for each fuzzy singleton r_s of R such that $r_s \not\subset F - annX$.

Let $a_l \subseteq F - ann r_s x_t$ for a fuzzy singleton r_s of R and $x_t \subseteq X$, then $a_l(r_s x_t) \subseteq 0_1$.

Consequently, $(a_l r_s)x_t \subseteq 0_1$. This implies that $a_l r_s \subseteq F - annx_t$, so $a_l r_s \subseteq F - annX$. But F-annX=F-annA,then $a_l r_s \subseteq F - annA$, thus F-annA is prime fuzzy ideal by [1, definition 2.1.1], since $r_s \not\subset F - annA$.

So $a_l \subseteq F - annA$, implies that $a_l \subseteq F - annX$, hence $a_l X \subseteq 0_1$, that is $a_l x_t \subseteq 0_1$ for all $x_t \subseteq X$, hence $F - ann r_s x_t \subseteq F - annx_t$ …(1)

Now, let $a_l \not\in\subseteq F - annx_t$ for a fuzzy singleton a_l of R and $x_t \subseteq X$.

By lemma 3.4, there exists fuzzy singleton r_s of R such that $0_1 \neq r_s x_t \subseteq A$, that is $r_s \not\subset F - annx_t$, so $a_l(r_s x_t) \subseteq 0_1$ which implies that $a_l \subseteq F - ann(r_s x_t)$, thus $F - annx_t \subseteq F - ann r_s x_t$ …(2)

From (1) and (2) we get F-annx_t=F-ann$r_s x_t$, thus X is a quasi prime fuzzy module by [1, theorem 2.1.9(3)].

Definition 3.8 [6]:

A fuzzy module X is called fuzzy torsion free if F-ann$x_t=0_1$ for any $x_t \subseteq X, x_t \neq 0_1$ where $F - annx_t = \{r_k : r_k \text{ fuzzy singleton of R}, r_k x_t \subseteq 0_1\}$.

Lemma 3.9:

Let R be an integral domain and let X be a torsion free fuzzy module of R-module M, then X is a quasi prime fuzzy module.

Proof:

Suppose that X is a torsion free fuzzy module. To prove 0_1 is quasi prime fuzzy submodule of X.

Let $a_l b_k x_t \subseteq 0_1$,for fuzzy singletons a_l,b_k of R and $x_t \subseteq X$.

If $b_k x_t \not\subset 0_1$, then $a_l \subseteq F - ann b_k x_t$, since $F - ann(b_k x_t) = F - ann y_\lambda$, where

$\lambda = \min\{k,t\}$ **and y=bx by [1, Remark 1.2.14] so** $a_l \subseteq F-anny_\lambda$. **But** $F-anny_\lambda = 0_1$ **because X is a torsion free, so** $a_l \subseteq 0_1$.

On the other hand, $0_1 \subseteq F-annX$, **which implies that** $a_l \subseteq F-annX$, **that is**
$a_l X \subseteq 0_1$, **by [6, definition 1.2.17], hence** $a_l x_t \subseteq 0_1$ **for all** $x_t \subseteq X$.

Thus, X is a quasi prime fuzzy module by [1,proposition 3.4.1].

4. Quasi Primary fuzzy submodule.

In this section we introduce the concept of quasi primary fuzzy submodule by fuzzifying the ordinary notion of primary submodule .

Definition 4.1 [5]:

A fuzzy submodule A of a fuzzy module X is called a primary fuzzy submodule if for each fuzzy singleton r_t of R and $a_k \subseteq X$ **such that** $r_t a_k \subseteq A$, **then either**

$$a_k \subseteq A \text{ or } r_t \subseteq \sqrt{(A:X)} = \{m_s \subseteq R, \textit{there exists } n \in Z^+, \textit{such that } m_s^n \subseteq (A:X)\}$$

,where
$(A:X) = \{r_t : r_t X \subseteq A, r_t \textit{ fuzzy singleton of } \mathbf{R}\}$

Recall that if A is a fuzzy submodule of a fuzzy module X of an R-module and I is a fuzzy ideal of R then (A:I) define as:
$(A:I) = \{a_k, a_k I \subseteq A, a_k \subseteq X\}$, **note that :If I=($r_t$),then:**
$(A:(r_t)) = \{a_k : r_t a_k \subseteq A, a_k \subseteq X\}$, **[8].**

Proposition 4.2:

Let A be a proper fuzzy submodule of a fuzzy module X, then the following statements are equivalent:

1. A is a primary fuzzy submodule of X.
2. (A:I) is a primary fuzzy submodule of X for each fuzzy ideal I of R.
3. (A:(r_t)) is a primary fuzzy submodule of X for each fuzzy singleton r_t of R.

Proof:

$1 \to 2$:

Suppose that $r_t x_k \subseteq (A:I)$ **for each** $x_k \subseteq X$ **and a fuzzy singleton r_t of R and I be a fuzzy ideal of R.It follows that** $a_l r_t x_k \subseteq A$ **for each fuzzy singleton a_l of I, then either** $a_l x_k \subseteq A$ **for each fuzzy singleton a_l of I or** $r_t^n \subseteq (A:X)$, $n \in Z^+$.

So, either $x_k \subseteq (A:I)$ or $r_t^n X \subseteq A$, **since** $A \subseteq (A:I)$, **by [8, theorem 3.3], hence** $r_t^n X \subseteq (A:I)$, **implies that**

$r_t^n \subseteq ((A:I):X)$, **it follows, either** $x_k \subseteq (A:I)$ or $r_t^n \subseteq ((A:I):X)$, **hence (A:I) is a primary fuzzy submodule of X .**

$2 \to 3$:

It is clear.

$3 \to 1$:

It follows easily by taking r=1 and t=1.

Recall that a proper submodule N ofan R-module M is said to be quasi primary submodule if [N:K] is a primary ideal of R if for each submodule K of M such that $N \subset K$ **,where [N:K]={r∈R:rk⊆N},[9].**

We shall fuzzify this concept as follows:

Definition 4.3 :

A proper fuzzy submodule A of a fuzzy module X is said to be quasi primary fuzzy submodule if (A:B) is a primary fuzzy ideal of R for each fuzzy submodule B of X such that $A \subset B$ **where**

$$(A:B) = \{r_t / r_t B \subseteq A, \textit{ for } \text{a fuzzy singleton } r_t \text{ of } R\}$$
.

Proposition 4.4:

Let A be a proper fuzzy submodule of a fuzzy module X, then A is a quasi primary fuzzy submodule of X if and only if $\sqrt{A:B} = \sqrt{A:r_t B}$ for each primary fuzzy submodule B of X and for each fuzzy singleton r_t of R such that $A \subset B$ and $r_t B \not\subset A$.

Proof:

It is clear that $\sqrt{A:B} \subseteq \sqrt{A:r_t B}$.

Now, let $a_k \subseteq \sqrt{A:r_t B}$ for all $r_t B \not\subset A$, hence $a_k^n \subseteq (A:r_t B)$ for some $n \in Z^+$ and so $a_k^n r_t B \subseteq A$, which implies that $a_k^n r_t \subseteq (A:B)$. But (A:B) is a primary fuzzy ideal of R and $r_t \not\subset (A:B)$, so $\left(a_k^n\right)^m \subseteq (A:B)$ for some $m \in Z^+$, thus $a_k^s \subseteq (A:B)$ where s=nm so that $a_k \subseteq \sqrt{A:B}$; that is $\sqrt{A:r_t B} \subseteq \sqrt{A:B}$ and hence $\sqrt{A:B} = \sqrt{A:r_t B}$.

Conversely, to prove (A:B) is a primary uzzy ideal of R for each $A \subset B$.

Suppose that $a_k r_t \subseteq (A:B)$ for a fuzzy singleton a_k and r_t of R. If $r_t \not\subset (A:B)$, then $a_k \subseteq (A:r_t B) \subseteq \sqrt{A:r_t B} = \sqrt{A:B}$ and hence A is a quasi primary fuzzy submodule of X.

Proposition 4.5:

Let A be a fuzzy submodule of a fuzzy module X, then A is a quasi primary submodule of X if and only if (A:I) is a quasi primary fuzzy submodule of X for each fuzzy ideal I of R.

Proof:

Suppose that A is a quasi primary fuzzy submodule of X.

By [8, theorem 3.3], (A:I) is a fuzzy submodule of X.

To prove (A:I) is a quasi primary fuzzy submodule of X, we must prove ((A:I):B) is a primary fuzzy ideal for each $(A:I) \subseteq B$.

$A \subseteq (A:I) \subset B$ by [8, lemma 2.16(1)].

Now, suppose that $a_k r_t \subseteq ((A:I):B)$ for fuzzy singleton r_t and a_k of R, then $a_k r_t B \subseteq (A:I), so\ a_k r_t BI \subseteq A$, suppose that $a_k \not\subset ((A:I):B)$, then $a_k BI \not\subset A$, hence $a_k r_t I \subseteq (A:B)$.

But, (A:B) is a primary fuzzy ideal by definition 4.3 and $a_k I \not\subset (A:B)$. Thus, $r_t^n \subseteq (A:B)$ for some $n \in Z^+$ and so $r_t^n B \subseteq A \subseteq (A:I)$, therefore $r_t^n \subseteq ((A:I):B)$ and hence (A:I) is a quasi primary fuzzy submodule of X.

The converse follows by taking $I = \chi_R$, where $\chi_R(a) = 1$ for all $a \in R$ and by similar to the proof of [6, proposition 2.1.16].

Recall that an R-module m is said to be a quasi primary module if annN is a primary ideal of R, for each non zero submodule N of M,[9].

Our attempt to fuzzify this concept is as follows:

Definition 4.6 :

A fuzzy module X of R –module M is said to be quasi primary fuzzy module if and only if F-annA=$(0_1:A)$ is a primary fuzzy ideal for each nonempty fuzzy submodule A of X .

Proposition 4.7:

A fuzzy module X is a quasi primary fuzzy module if and only if 0_1 is a quasi primary fuzzy submodule of X.

Proof:

Suppose that X is a quasi primary fuzzy module.

To prove 0_1 is a quasi primary fuzzy submodule of X, since X is a quasi primary fuzzy submodule, then F-annA is a primary fuzzy ideal for each nonempty fuzzy submodule A of X.Thus $(0_1:A)$ is primary fuzzy ideal and hence 0_1 is quasi primary fuzzy submodule of X.

Conversely, if 0_1 is quasi primary fuzzy submodule of X, to prove X is a quasi primary fuzzy module. Since 0_1 is a quasi primary fuzzy submodule, then $(0_1:A)$ is a primary fuzzy ideal for each nonempty fuzzy submodule.

But $(0_1:A)$=F-annA, so X is a quasi primary fuzzy R-module.

REFRENCES:

[1] Hatem Y.K.,(2001), "Fuzzy Quasi –prime Modules and Fuzzy Quasi –prime submodules ", M.Sc. thesis, University of Baghdad, college of Education Ibn-AL- Haithem ,(69)p.

[2] Martines L.,(1996), "Fuzzy modules over fuzzy rings in connection with Fuzzy ideal of Fuzzy ring ". J.Fuzzy Mathematics, Vo1.4:843-857.

[3] Liu W. J,(1982),"Fuzzy invariant subgroups and Fuzzy ideals", fuzzy sets and systems,Vol.8:133-139.

[4] Nanda S. (1989),"Fuzzy Modules over Fuzzy rings ", Bull. coll. Math. Soc., Vo1. 18 :197-200.

[5] Mashinchi M. and Zahedi M.M.,(1992)" On L-Fuzzy primary submodules ", Fuzzy sets and systems, Vol.49:231-236.

[6] Jari R.H.(2001)," prime Fuzzy submodules and prime Fuzzy modules ", M.Sc. Thesis, University of Baghdad, college of Education, Ibn –AL-Haithem,(64)p.

[7] Mukhrjie T.K.,(1989)" Prime Fuzzy Ideals in Rings", Fuzzy Sets and Systems Vol. 32:337-341.

[8] Zahedi M.M.,(1992),"On L-Fuzzy Residual Quotient Modules", Fuzzy sets and systems, Vol.51:333-344.

[9] Abdul-al-Kalik A.J.,(2005),"Primary Modules", M.Sc. thesis, University of Baghdad,college of Education Ibn- AL-Haithem,(77)p.

On Quasi Prime Fuzzy Submodules and Quasi Primary Fuzzy Submodules

ربيع هادي جاري حسن كريم حسن
قسم الرياضيات - كلية التربية - جامعة ذي قار

الخلاصة:

في هذا البحث اعطينا بعض النتائج عن المقاسات الجزئية شبة الاولية الضبابية ,كذلك درسنا مفهوم المقاسات الجزئية شبه الابتدائية الضبابية .