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# THE BROKEN CIRCUIT COMPLEX
# AND
# THE HYPERSOLVABLE PARTITION COMPLEX

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## Abstract

In this paper we construct the broken circuit complex of a hypersolvable $r$-arrangement A by using the hypersolvable partition analogue and the hypersolvable ordering which respects the hypersolvable structure .We used the minimal informations that encoded in the intersection lattice pattern up to codimension two to define an isomorphism between the broken circuit complexes of any two supersolvable arrangements which enable us to produce a comparison between the structures of the broken circuit complexes of a hypersolvable arrangement and their Jambu's-Papadima's deformed supersolvable arrangements.

## Introduction

Let $\mathrm{A} = \{H_1,...,H_n\}$ be a complex hyperplane $r$-arrangement with complement $M(\mathrm{A}) = C^r \setminus \bigcup_{i=1}^{n} H_i$) we refer the reader to ]6 [as a general reference .(Many basic facts about the linear arrangements and their intersection poset $L(\mathrm{A})$) which reverse by inclusion, )i.e . $X \le Y \Leftrightarrow Y \subseteq X$ (and ranked by $rk(X) = \mathrm{codim}(X)$(, are best understand from the more general viewpoint of the matroid theory .An arrangement matroid is a pair $\mathrm{M}_{\mathrm{A}} = (\mathrm{A}, \Delta)$, by letting A be the set of all vertices of the simplicial complex $\Delta$, where $\Delta$ be the collection of all independent subarrangements of $\mathcal{A}$ .The" broken circuit complex)"or BC-complex( of $M_{\mathcal{A}}$ is denoted by $BC_{\trianglelefteq}(M_{\mathcal{A}})$ and defined to be;

$$BC_{\trianglelefteq}(M_{\mathcal{A}}) = \{B \subseteq \mathcal{A} \mid B \text{ contains no broken circuit}\}.,$$

where a broken circuit of a matroid $M_{\mathcal{A}}$ with respect to an ordering $\trianglelefteq$ on the hyperplanes of $\mathcal{A}$, is a subset $\overline{X} = X \setminus H$ of a minimal dependent set $X$) with respect to the inclusion(, which is called a circuit and $H$ is the minimal element of $X$ via $\trianglelefteq$.

Jambu and Papadima in )1998, ]3 ([and )2002, ]4 ([introduced the hypersolvable class of arrangements as a generalization of the supersolvable Stanely class )1972, ]9 ([by using the collinear relations that encoded in the lattice intersection pattern up to codimension two $\Lambda_2(\mathcal{A}) = \{B \subseteq \mathcal{A} \,\|\, B| \le 3\}$. In order to control and study the lattice intersection of a hypersolvable arrangement $\mathcal{A}$, Ali in )2007, ]1 ([drived a natural partition $\Pi$ of $\mathcal{A}$ from the hypersolvable analogue )definition )1.2 .((She called $\Pi$ a hypersolvable partition, denoted it by Hp and she ordered the hyperlanes of $\mathcal{A}$ by a hypersolvable ordering )definition 1.5 (which respects the hypersolvable analogue .The existences of such partition forms a sufficient condition to any central arrangement to be hypersolvable arrangement .This paper has two purposes .The first is to construct the broken circuit complex $BC_{\trianglelefteq}(M_{\mathcal{A}})$ via the hyper solvable ordering as an application of Ali's study in ]1 .[

Björner and Ziegler in )theorem )2.8(, ]2([, answered an essential question; where does the broken circuit complex $BC_{\trianglelefteq}(M_{\mathcal{A}})$ factored completely )definition )1.3 ((in general? To switched our attention that we need a sutible linear order on the hyperplanes of a supersolvable arrangement $\mathcal{A}$ to factor the broken circuit complex $BC_{\trianglelefteq}(M_{\mathcal{A}})$ completely into a multiple join of discrete 0-dimensional subcomplexes .They gave us an impression to express the hypersolvable ordering in theorem )3.1 (as our choice of such linear order indeed it was drived from the supersolvable structure.

Jambu and Papadima in ] )3-4 ([defined a vertical deformation $\{\tilde{\mathcal{A}}_t\}_{t\in C}$ of a hypersolvable $r$-arrangement $\mathcal{A}$ which is not supersolvable such that for each $t \in C \setminus \{0\}$, $\tilde{\mathcal{A}}_t$ is supersolvable and $\mathcal{A}$ has with $\tilde{\mathcal{A}}_t$ the same $\Lambda_2$ .For a supersolvable arrangement all the higher homotopy groups of the complement are vanished and such arrangements are called $K(\pi = \pi_1(M(\mathcal{A})),1)$ arrangements, where $\pi_1(M(\mathcal{A}))$ is the fundamental group of $M(\mathcal{A})$ .Papadima and Suciu ]7[ used the one parameter family $\{\tilde{\mathcal{A}}_t\}_{t\in C}$ of Jambu and Papadima for a hypersolvable arrangement $\mathcal{A}$ which is not supersolvable )fiber-type(, to show that the dimension of the first non vanishing higher homotopy group is;

$$p(M(\mathcal{A})) = sup\{k \mid P(H^*(M(\mathcal{A})),s) \equiv_{mod\, j} P(H^*(M(\tilde{\mathcal{A}}_1)),s), \forall j \le k\},$$

where $P(H^*(M(\mathcal{A})),s)$ and $P(H^*(M(\tilde{\mathcal{A}}_1)),s)$ are the Poincaré polynomials of the cohomological rings $H^*(M(\tilde{\mathcal{A}}_1))$ and $H^*(M(\tilde{\mathcal{A}}_1))$ respectively. Ali in ]1 [showed a conjecture of $p(M(\mathcal{A}))$ in order to produce a connection between the dimension of the first non vanishing higher homotopy group of a hypersolvable arrangement and the structure of it's no broken circuit bases with respect the hypersolvable ordering ) see definition )3.3 .((The second purpose of this paper is to produce a comparison between the structures of the broken circuit complexes of a hypersolvable arrangement A which is not supersolvable )i.e .with respect to any ordering $\trianglelefteq$ on the hyperplanes of A , it's broken circuit complex can not be factored completely( and their Jambu's-Papadima's deformed fiber-type arrangements $\tilde{\mathrm{A}}_t$ , $t \in C \setminus \{0\}$ .We found that $p(M(\mathrm{A}))$ forms the rank of the last level in the intersection lattice that is invariant under Jambu's-Papadima's deformation .

The sections in this paper are devoted to serve our goal. Section one contains a brief summary of the notions "matroid "and "the broken circuit complex "in oreder to introduce the basic defintions that we need .In section two we review the hypersolvable partition and the hypersolvable ordering of a hypersolvable arrangement .Where section three is devoted to ordered the hyperplanes of a hypersolvable arrangement by the hypersolvable ordering in order to give the broken circuit complex a geometric structure . For the supersolvable subclass of the hypersolvable class we found that the broken circuit factored completely with respect to the hypersolvable ordering )theorem )3.1 .((Where for the hypersolvable subclass of arrangement which are not supersolvable, we study their broken circuit complexes in (theorem )3.5.))

## 1 . Matroids and the broken circuit complexes

Let $\Delta$ be a "finite "simplicial complex with vertex set $\mathrm{A} = \{v_1,...,v_n\}$ .We call the elements of $\Delta$ the faces of $\Delta$ .If the maximal face of $\Delta$ has $b$ elements, then we say $\dim\Delta = \delta = b-1$ .If every maximal face of $\Delta$ has dimension $\delta$ , then $\Delta$ is called pure .The $f$ -vector of $\Delta$ is a vector of integers $f = (f_0, f_1,..., f_\delta)$, where for $0 \le k \le \delta$ , $f_k$ is defined to be the number of the faces of $\Delta$ have $k+1$ elements .Notice that, $f_0 = |\mathrm{A}| = n$ .Define for a positive integers $m$ ,

$$H(\Delta, m) = \sum_{k=0}^{\delta} f_k \begin{pmatrix} m-1 \\ k \\ \end{pmatrix}.$$

Also define $H(\Delta,0) = 1$ .Define the $h$-vector of integers $h = (h_1,...,h_b)$ of $\Delta$ as follows :

$$\sum_{m=0}^{\infty} H(\Delta, m)x^m = \frac{(1+h_1x+\cdots+h_bx^b)}{(1-x)^b}.$$

The $h$ -vector of $\Delta$ is determined by its $f$ -vector.

Let $K$ be a field and let $A = K[x_1,...,x_n]$ be the polynomial ring over $K$ whose variables are the vertices of $\Delta$ .Let $I_\Delta$ be the homogenous ideal of $A$ generated by all squarefree monomials $x_{i_1}...x_{i_k}$ such that $\{x_{i_1}...x_{i_k}\}$ is non-faces of $\Delta$, i.e . $I_\Delta$ is generated by the "minimal "non-faces of $\Delta$ .We call the ring $A_\Delta = A/I_\Delta$ a standard $K$-algebra .As a graded algebra $A_\Delta = \sum_{m=0}^{\infty} A_\Delta^m$, define the Hilbert function $H(A_\Delta, m)$ of $A_\Delta$ by $H(A_\Delta, m) = \dim(A_\Delta^m)$ and the Krull dimension of $A_\Delta$ which is denoted by $\dim(A_\Delta)$ is one more than the maximal integer m such that $H(A_\Delta, m) \ne 0$ .If;

$$0 \to M_h \to M_{h-1} \to \cdots \to M_0 \to A_\Delta \to 0,$$

is the minimal finite free resolution of $A_\Delta$, then for $0 \le i \le h$ we define the $j^{th}$-Betti number of $A_\Delta$ to be $\beta_i(A_\Delta) = \beta_i = rk(M_i)$ .The integer $h$ which represent the largest integer $i$ such that

$\beta_i \neq 0$ is called the homological dimension of $A_\Delta$ denoted by $hd_{A_\Delta}$ .If $hd_{A_\Delta} = n - \dim(A_\Delta)$, we call $A_\Delta$ a Cohen-Macaulay ring and $\beta_{hd_{A_\Delta}}$ is called the type of $A_\Delta$ .We refer to ]8 [and ]10 [for more details.

**Definition 1.1** A" finite "matroid is a pair $\mathrm{M} = (\mathrm{A}, \Delta)$, where $\mathrm{A}$ is a finite set and $\Delta$ is a collection of subsets of $\mathrm{A}$ , satisfying the following axioms :

1 . $\Delta$ is a nonempty simplicial complex, i.e . $\Delta \neq \varnothing$ and if $\Delta' \in \Delta$ and $\Delta'' \subset \Delta'$, then $\Delta'' \in \Delta$.

2 .Every induced subcomplex of $\Delta$ is pure, i.e .if $\mathrm{B} \subseteq \mathrm{A}$ , the maximal elements of $\Delta \bigcap 2^{\mathrm{B}}$ have the same cardinality, where $2^{\mathrm{B}} = \{\mathrm{X} \subseteq \mathrm{A} \mid \mathrm{X} \subseteq \mathrm{B}\}$ .

The elements of $\Delta$ are called independent sets and we write $v \in \mathrm{M}$ to mean $v \in \mathrm{A}$ .We call $\Delta$ a $G$-complex .

Two matroids $\mathrm{M}_1 = (\mathrm{A}_1, \Delta_1)$ and $\mathrm{M}_2 = (\mathrm{A}_2, \Delta_2)$ are isomorphic if there exists a bijection $\psi : \mathrm{A}_1 \xrightarrow{\sim} \mathrm{A}_2$ such that $\{v_1, ..., v_k\} \in \Delta_1$ if, and only if, $\{\psi(v_1), ..., \psi(v_k)\} \in \Delta_2$ .

A circuit $\mathrm{X} \subseteq \mathrm{A}$ is a minimal dependent set, i.e . $\mathrm{X}$ is not independent but becomes independent when we remove any point from it .If $\mathrm{B} \subseteq \mathrm{A}$ , we define the rank of $\mathrm{B}$ by;

$$rk(\mathrm{B}) = \max\{|\mathrm{B}'| \mid \mathrm{B}' \subseteq \mathrm{B} \text{ and } \mathrm{B}' \in \Delta\}.$$

In particular, $rk(\varnothing) = 0$ .We define the rank of matroid $\mathrm{M}$ itself by $rk(\mathrm{M}) = rk(\mathrm{A})$ .A maximal subset $\mathrm{B} \subseteq \mathrm{A}$ of rank $k$ is said to be a $k$-flat of $\mathrm{M}$ .Observe that if $\mathrm{B}$ and $\mathrm{B}'$ are flats of $\mathrm{M}$ , then so is $\mathrm{B} \bigcap \mathrm{B}'$ .We can define the closure $\overline{\mathrm{B}}$ a subset $\mathrm{B} \subseteq \mathrm{A}$ to be the smallest flat containing $\mathrm{B}$, i.e . $\overline{\mathrm{B}} = \bigcap_{flats \mathrm{B}' \supseteq \mathrm{B}} \mathrm{B}'$ .Define the $L(\mathrm{M})$ to be the poset of flats of $\mathrm{M}$ ordered by inclusions .Since $L(\mathrm{M})$ has a top element $\mathrm{A}$ , then $L(\mathrm{M})$ is a lattice called the lattice of flats of $\mathrm{M}$ .Notice that, $L(\mathrm{M})$ has a unique minimal element is $\hat{0} = \varnothing$ .We define the characteristic polynomial $\chi_{\mathrm{M}}(t)$ of $\mathrm{M}$ , by;

$$\chi_{\mathrm{M}}(t) = \sum_{X \in L(\mathrm{M})} \mu(\hat{0}, X) t^{r - rk(X)},$$

where $r = rk(\mathrm{M})$ and $\mu$ is the Möbius function of $L(\mathrm{M})$ and $|\mu(\hat{0}, X)|$ is represent the length of the maximal chain from minimal flat $\hat{0}$ into $X$ of $L(\mathrm{M})$ .Since $\Delta$ is a $G$-complex, then for $0 \leq k \leq rk(\mathrm{M})$;

$$\beta_k(A_\Delta) = \beta_k = \sum_{flats X \in L_k(\mathrm{M})} |\mu(\hat{0}, X)|.$$

The $A_\Delta$ is a Cohen-Macaulay ring and the Hilbert function $H(A_\Delta, m) = H(\Delta, m)$ with homological dimension $hd_{A_\Delta} = rk(\mathrm{M})$ and of type $\beta_{rk(\mathrm{M})}(A_\Delta)$ .

**Definition 1.2** A broken circuit of a matroid $\mathrm{M}$ with respect to an ordering $\trianglelefteq$ of $\mathrm{A}$ , is a set $\overline{\mathrm{X}} = \mathrm{X} \backslash v$, where $\mathrm{X}$ is a circuit and $v$ is the minimal element of $\mathrm{X}$ via $\trianglelefteq$ .The broken circuit complex )or BC-complex (denoted by $BC_{\trianglelefteq}(\mathrm{M})$ is defined to be;

$$BC_{\trianglelefteq}(\mathrm{M}) = \{\mathrm{B} \subseteq \mathrm{A} \mid \mathrm{B} \text{ contains no broken circuit}\}.$$

For $0 \le k \le \delta$, set;

$$BC^k_{\trianglelefteq}(\mathrm{M}) = \{\mathrm{B} \subseteq \mathrm{A} \mid \mathrm{B} \text{ contains no broken circuit and } |\mathrm{B}| = k+1\}.$$

Notice that, Rota in ]8 [gave another representation of $|\mu(\hat{0}, X)|$ as the number of all the maximal no broken circuits of $\mathrm{M}_{\mathrm{A}}$ contained in the flat $X$ .That is, if $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, ..., f_{\delta}^{\Delta})$ is the $f$-vector of $BC_{\trianglelefteq}(\mathrm{M})$, then $f_k^{\Delta} = |BC^k_{\trianglelefteq}(\mathrm{M})| = \beta_{k+1}(A_{\Delta})$ and;

$$\chi_{\mathrm{M}}(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta},$$

where $r = rk(\mathrm{M})$ and $f_{-1}^{\Delta} = 1$ .Notice that, $h_r = \beta_{rk(\mathrm{M})}(A_{\Delta}) = f^{\Delta}_{rk(\mathrm{M})-1}$ is the type of the Cohen-Macaulay ring $A_{\Delta}$ .

**Definition 1.3** ]2 [Let $\Delta$ be a simplicial complex of dimension $r-1$ with finite vertex set $\mathrm{A}$ .We say that $\Delta$ factors if $\mathrm{A}$ has a partition $\Pi = (\Pi_1, \Pi_2)$ such that $\Delta = \Delta_1 * \Delta_2$, where $\Delta_i = \Delta|_{\Pi_i} = \{S \in \Delta \mid S \subseteq \Pi_i\}$ is the restriction of $\Delta$ to $\Pi_i$ )$i = 1,2$(, and the join of $\Delta_1$ and $\Delta_2$ is $\Delta_1 * \Delta_2 = \{S_1 \bigcup S_2 \mid S_1 \in \Delta_1 \text{ and } S_2 \in \Delta_2\}$ .We say that $\Delta$ factors completely if $\mathrm{A}$ has a partition $\Pi = (\Pi_1, ..., \Pi_r)$ into $r$ nonempty sets, such that $\Delta$ is a multiple join of the induced subcomplexes $\Delta = \Delta_1 * \cdots * \Delta_r$ )as above(, where $\Delta_i$ are discrete 0-dimensional, i.e . $\Delta_i = \{\varnothing\} \bigcup \{\{v\} \mid v \in \Pi_i\}$, for $1 \le i \le r$.

**Definition 1.4** )***A hyperplanes arrangements and their broken circuit complexes***(Let $\mathrm{A}$ be a central $r$-arrangement of hyperplanes over $C$, i.e . $\mathrm{A} = \{H_1, ..., H_n\}$, where $H_i$ are linear hyperplanes of $C^r$, $\bigcap_{i=1}^n H_i \ne \varnothing$ and $\mathrm{codim}(\bigcap_{i=1}^n H_i) = r$ .Define the complement $M(\mathrm{A}) = C^r \setminus \bigcup_{i=1}^n H_i$ and $L(\mathrm{A})$ to be the lattice intersections of the hyperplanes of $\mathrm{A}$ reverse by inclusion, )i.e . $X \le Y \Leftrightarrow Y \subseteq X$ (and ranked by $rk(X) = \mathrm{codim}(X)$.

Define a matroid $\mathrm{M}_{\mathrm{A}} = (\mathrm{A}, \Delta)$, on $\mathrm{A}$ by letting $\Delta$ to be the collection of all independent subarrangements of $\mathrm{A}$ .Notice that, $L(\mathrm{A}) \equiv L(\mathrm{M})$ .Let

$$BC_{\trianglelefteq}(\mathrm{M}) = \{\mathrm{B} \subseteq \mathrm{A} \mid \mathrm{B} \text{ contains no broken circuit}\},$$

be the BC-complex of $\mathrm{M}_{\mathrm{A}}$ .Then

$$\chi_{\mathrm{M}}(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta},$$

where $r = rk(\mathrm{A}) = \delta + 1$ and $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, ..., f_{\delta}^{\Delta})$ be the $f$-vector of $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ and $f_{-1} = 1$ .Notice that, $h_r = \beta_{rk(\mathrm{M}_{\mathrm{A}})}(A_{\Delta}) = f_{r-1}^{\Delta}$ is the type of the Cohen-Macaulay ring $A_{\Delta}$ and it has a minimal free resolution,

$$0 \to M_r \to M_{r-1} \to \cdots \to M_0 \to A_{\Delta} \to 0,$$

where for $0 \le k \le r$, $rk(M_k) = f_{k-1}^{\Delta}$.

## 2. A hypersolvable partition of an arrangement

**Definition 2.1** ]6 [A partition $\Pi=(\Pi_1,...,\Pi_\ell)$ of $\mathrm{A}$ is said to be independent if the resulting $\ell$-hyperplanes $H_j \in \Pi_j$, $1\le j\le \ell$ are independent .Call $S=\{H_{i_1},...,H_{i_k}\}$ a $k$-section of $\Pi$ if for each $1\le j\le k$, $H_{i_j}\in \Pi_{m_j}$ for some $1\le m_1<\cdots<m_k\le \ell$ .Notice that if $\Pi$ is independent, then all it's $k$-sections are independent .Let;

$S^k_\Pi(\mathrm{A})=\{S\subseteq \mathrm{A} \mid S \text{ is a } k\text{-section of } \Pi\}$, for $1\le k\le \ell$,

and $S_\Pi(\mathrm{A})=\bigcup_{k=1}^{\ell} S^k_\Pi(\mathrm{A})$ .We call $\Pi$ a factorization of $\mathrm{A}$, if it is independent and for each $X\in L_k(\mathrm{A})$, the induced partition $\Pi_X=(\Pi^1_X,...,\Pi^k_X)$ of $\mathrm{A}_X=\{H\in \mathrm{A} \mid X\subseteq H\}$, contains a singleton block, where for $1\le j\le k$, $\Pi^j_X=\Pi_m \cap \mathrm{A}_X\ne \phi$ for some $1\le m\le \ell$ .

**Definition 2.2**] 1 [A partition $\Pi=(\Pi_1,...,\Pi_\ell)$ of $\mathrm{A}$ is said to be hypersolvable with length $\ell(\mathrm{A})=\ell$ and denoted by Hp, if $|\Pi_1|=1$ and for fixed $2\le j\le \ell$, the block $\Pi_j$ satisfy the following properties :

)***j*-closedness property of $\Pi_j$ :**(For $H_1,H_2\in \Pi_1\cup\cdots\cup\Pi_j$, there is no $H\in\Pi_{j+1}\cup\cdots\cup\Pi_\ell$ such that $rk(H_1,H_2,H)=2$ .

)***j*-completeness property of $\Pi_j$ :**(For $H_1,H_2\in\Pi_j$, there is $H\in\Pi_1\cup\cdots\cup\Pi_{j-1}$ such that $rk(H_1,H_2,H)=2$ .From $j-1$-closeness property of $\Pi_{j-1}$, the hyperplane $H$ must be unique we denoted by $H_{1,2}$ .

)***j*-solvability property of $\Pi_j$ :**(For $H_1,H_2,H_3\in\Pi_j$, either $H_{1,2},H_{1,3},H_{2,3}\in\Pi_1\cup\cdots\cup\Pi_{j-1}$ are equal or $rk(H_{1,2},H_{1,3},H_{2,3})=2$ .

For $1\le j\le \ell$, let $d_j=|\Pi_j|$ .The vector of integers $d=(d_1,...,d_\ell)$ is called the $d$-vector of $\Pi$ and we define the rank of the blocks of $\Pi$ as $rk(\Pi_j)=rk(\bigcap_{H\in\Pi_1\cup\cdots\cup\Pi_j} H)$ .We call $\Pi_j$ singular if $rk(\Pi_j)=rk(\Pi_{j-1})$ and we call it non singular otherwise .We call a Hp $\Pi$ is supersolvable if it is independent .Observe that $rk(\Pi_{j-1})\le rk(\Pi_j)$ in general and if $\ell\ge 3$, then every $\Pi_{i_1},\Pi_{i_2},\Pi_{i_3}\in\Pi$ are independent .

**Definition 2.3**] 1 [$\mathrm{A}$ is said to be hypersolvable if it has a hypersolvable partition and we call it supersolvable if it has a supersolvable partition .

**Definition 2.4** Let $\mathrm{A}$ be a hypersolvable arrangement with Hp $\Pi=(\Pi_1,...,\Pi_\ell)$ .For $1\le j\le \ell$, partitioned $\Pi_j$ into two blocks as; put $\Pi_{j*1}=\{H_{i_1},...,H_{i_k}\}\subseteq\Pi_j$ such that $rk(H_{i_1},...,H_{i_k})=2$ and $\Pi_{j*2}=\Pi_j\setminus\Pi_{j*1}$ .Define the induced hypersolvable ordering $\trianglelefteq$ of $\mathrm{A}$ as :

1 .If $H\in\Pi_i$ and $H'\in\Pi_j$ such that $i<j$, put $H\trianglelefteq H'$ .If $H\in\Pi_i$ and $H'\in\Pi_j$ such that $i<j$, put $H\trianglelefteq H'$ .

2 .For $2 \le j \le \ell$, give the hyperplanes of $\Pi_{j*1}$ an arbitrary ordering such that if $H_1, H_2, H_3 \in \Pi_j$ with $rk(H_1, H_2, H_3) = 3$, put $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1,i_2} \trianglelefteq H_{i_1,i_3} \trianglelefteq H_{i_2,i_3}$, where $\{H_{i_1}, H_{i_2}, H_{i_3}\} = \{H_1, H_2, H_3\}$ .

Notice that the induced hypersolvable ordering need not be unique of $\mathrm{A}$ , since our choice of $\Pi_{j*1}$ need not be unique.

**Definition 2.5**] 6 [A $r$-arrangement $\mathrm{A}$ is said to be $\ell$-generic) $\ell > r$ (, if every subarrangement $\mathrm{B} \subseteq \mathrm{A}$ with $|\mathrm{B}| = r$ is linearly independent .

Observe that, the $\ell$-generic $r$-arrangement $\mathrm{A}$ is a hypersolvable arrangement has Hp $\Pi$ with exponent vector $d = (1,..,1)$, but the converse need not to be true in general and ]5 [included a counterexample.

## 3 The BC-complex of a hypersolvable arrangement and The hypersolvable partition

From now on we concern the assumpption : $\mathrm{A}$ is a hypersolvable $r$-arrangement with Hp $\Pi = (\Pi_1, ..., \Pi_\ell)$ and $d$-vector $d = (d_1, ..., d_\ell)$, where $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ be the BC-complex of the matroid $\mathrm{M}_{\mathrm{A}}$ via a hypersolvable ordering with $f$-vector $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, ..., f_{\delta}^{\Delta})$ . That is, we shall give the no broken circut subarrangements the degree lexicographic )DegLex (order with respect the hypersolvable ordering. Let $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})\big|_i = \{S \in BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}}) \mid S \subseteq \Pi_i\}$ be the restriction of $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ to $\Pi_i$, for $1 \le i \le \ell$ .

**Definition 3.1** Recall definition )2.1 .(Let; $S_{\Pi}^{k}(\mathrm{A}) = \{S \subseteq \mathrm{A} \mid S \text{ is a } k\text{-section of } \Pi\}$, for $1 \le k \le \ell$, be a discrete 0-dimensional simplicial subcomplex of $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$, for $1 \le i \le \ell$ .Let $S_{\Pi}(\mathrm{A}) = S_{\Pi}(\mathrm{A})\big|_1 * \cdots * S_{\Pi}(\mathrm{A})\big|_\ell$ be a multiple join of the induced subcomplexes $S_{\Pi}(\mathrm{A})\big|_1, \ldots, S_{\Pi}(\mathrm{A})\big|_\ell$ )as in definition )3.2((, i.e. $S_{\Pi}(\mathrm{A}) = \bigcup_{k=1}^{\ell} S_{\Pi}^{k}(\mathrm{A})$. We call $S_{\Pi}(\mathrm{A})$, a hypersolvable partition complex of the matroid $\mathrm{M}_{\mathrm{A}}$ via a hypersolvable ordering .In general, $S_{\Pi}(\mathrm{A})$ need not be a subcomplex of the $G$-complex $\Delta$ of the matroid $\mathrm{M}_{\mathrm{A}}$ .

**Lemma 3.1** For $1 \le i \le \ell$, $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})\big|_i = S_{\Pi}(\mathrm{A})\big|_i$.

*Proof:* It is known that, "In a hypersolvable arrangement $\mathrm{A}$ every $k$-NBC base of $\mathrm{A}$ forms a $k$-section of $\Pi$, $1 \le k \le r$", )proposition )3.1.22(, ]1([ and "$S \subseteq \mathrm{A}$ is a 2-NBC base of $\mathrm{A}$ if, and only if, $S$ is a 2-section of $\Pi$", )theorem )3.2.1(, ]1 .([Thus $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})\big|_{\Pi_i} = \{S \in BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}}) \mid S \subseteq \Pi_i\} = \{\varnothing\} \bigcup \{\{H\} \mid H \in \Pi_i\}$ are discrete 0-dimensional which completes the proof .

Bj ö rner and Ziegler in )theorem )2.8(, ]2([, answered an essential question; Is the broken circuit complex $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ factored completely in general? To switched our attention that the existence of a sutible linear order on the hyperplanes of a supersolvable arrangement $\mathrm{A}$ which respects the supersolvable structure forms a necessary and sufficient condition on the broken circuit complex $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ to factor completely and they gave us an impression to illustrate the hypersolvable ordering by the following theorem as our choice of the linear ordering satisfied theorem )2.8 (in ]2[, indeed it was drived from the supersolvable structure.

**Theorem 3.1** $\mathcal{A}$ is supersolvable if, and only if ; $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) = S_{\Pi}(\mathcal{A})$ ,i.e . $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ factors completely into $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) = BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})\big|_1 * \cdots * BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})\big|_r$ via the hypersolvable order . Therefore, for $1 \le k \le \delta+1 = r = \ell$; $\quad f_{k-1}^{\Delta} = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k}$ ,

i.e .the $f$ -vector of $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ is completely determined by the $d$ -vector of $\mathcal{A}$ .Notice that, $h_r = f_{r-1}^{\Delta} = d_2 ... d_\ell$ is the type of the Cohen-Macaulay ring $A_\Delta$ and it has a minimal free resolution, $0 \to M_r \to M_{r-1} \to \cdots \to M_0 \to A_\Delta \to 0$, where for $0 \le k \le r$ , $rk(M_k) = f_{k-1}^{\Delta}$ .

*Proof* .By )theorem )3.2.6(, ]1 ([," $\mathcal{A}$ hypersolvable $r$ -arrangement $\mathcal{A}$ is supersolvable if, and only if, every $k$ -section of $\Pi$ forms a $k$ -NBC-base of $\mathcal{A}$ via the hypersolvable order, $1 \le k \le r$ ", Thus, $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) = S_{\Pi}(\mathcal{A})$. and $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})\big|_{\Pi_i} = \{\varnothing\} \bigcup \{\{H\} \mid H \in \Pi_i\}$ is discrete 0-dimensional, for $1 \le i \le r$ .That is $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) = BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})\big|_{\Pi_1} * \cdots * BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})\big|_{\Pi_r}$ is completely factored .From definition )1.2(, for $1 \le k \le \delta+1 = r = \ell$;

$$f_{k-1}^{\Delta} = | BC_{\trianglelefteq}^{k}(\mathrm{M}_{\mathcal{A}}) | = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k} \text{ and,}$$

$f_{r-1}^{\Delta} = d_2 ... d_\ell$ .In fact we compute the $k^{th}$-Betti number of $A_\Delta$ which is enough to construct the minimal finite free resolution of $A_\Delta$ which is a Cohen-Macaulay ring with type $\beta_{hd_{A_\Delta}} = f_{r-1}^{\Delta}$

The important point to note here is that $S_{\Pi}(\mathcal{A})$ is a simplicial subcomplex of $\Delta$ if, and only if, $\mathcal{A}$ is supersolvable .That caused by these sections of $\Pi$ which is not independent in case of $\mathcal{A}$ is non- supersolvable .

**Example 3.1** Let $\mathcal{A}$ be a supersolvable 4 -arrangement with defining polynomial;

$Q(\mathcal{A}) = z(y-x+z)(y+x+z)(y+3z+w)(y+2z+w)(y+w)(y-z+w)$ .

We give the hyperplanes of $\mathcal{A}$ , the hypersolvable ordering as follows;

$H_1 = \ker(z)$ , $H_2 = \ker(y-x+z)$ , $H_3 = \ker(y+x+z)$ , $H_4 = \ker(y+3z+w)$ , $H_5 = \ker(y+2z+w)$ , $H_6 = \ker(y+w)$ and $H_7 = \ker(y-z+w)$ ,with respect the HP $\Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4) = (\{H_1\}, \{H_2\}, \{H_3\}, \{H_4, H_5, H_6, H_7\})$ which has a $d$ -vector, $d = (1,1,1,4)$ .For simplicity we write $i$ instead of $H_i$ , for each $1 \le i \le 7$ .The matroid $\mathrm{M}_{\mathcal{A}} = (\mathcal{A}, \Delta)$ , is defined on $\mathcal{A}$ by letting $\Delta = \bigcup_{k=0}^{3} \Delta_k$ , where; $\Delta_0 = \{1, \ldots, 7\}$ , $\Delta_1 = \{\{i, j\} \mid 1 \le i < j \le 7\}$ , $\Delta_2 = \{\{i, j, k\} \mid 1 \le i < j < k \le 7\} \setminus \{\{1, i, j\} \mid 4 \le i < j \le 7\}$ and $\Delta_3 = \{\{i, j, k, p\} \mid 1 \le i < j < k < p \le 7\} \setminus \{\{i, j, k, p\} \mid 1 \le i \le 3 \text{ and } 4 \le j < k < p \le 7\}$ .

That is the $f$-vector of $\Delta$ is $f=(7,21,25,10)$ and by simple calculations $H(\Delta,0)=1$, $H(\Delta,1)=7$, $H(\Delta,2)=28$, $H(\Delta,3)=74$, $H(\Delta,4)=155$, ...and the $h$-vector of $\Delta$ is $h=(1,3,6,1,0)$ since;

$$\sum_{m=0}^{\infty} H(\Delta,m)x^m = \frac{(1+3x+6x^2+x^3)}{(1-x)^4}.$$

By applying theorem )4.1(,

$$BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}(\mathrm{A});$$

and the $f$-vector of $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ with respect the hypersolvable ordering is $f^{\Delta}=(7,15,13,4)$ which is completely determined from the $d$-vector of $\Pi$.

Let $K$ be a field and let $A=K[H_1,...,H_7]$ be the polynomial ring over $K$ whose variables are the vertices of $\Delta$ .Let $I_{\Delta}$ be the homogenous ideal of $A$ generated by the "minimal "non-faces of $\Delta$ .The standard $K$-algebra $A_{\Delta}=\sum_{m=0}^{\infty} A_{\Delta}^m$, of $\Delta$, as a graded algebra is determined by the Hilbert function of $A_{\Delta}$ as $H(A_{\Delta},m)=\dim(A_{\Delta}^m)=H(\Delta,m)$ and the minimal finite free resolution of $A_{\Delta}$;

$$0 \to M_4 \to M_3 \to M_2 \to M_1 \to M_0 \to A_{\Delta} \to 0,$$

is completely determined by $f^{\Delta}$, since the Betti numbers of $A_{\Delta}$ are $\beta_0(A_{\Delta})=1$, $\beta_1(A_{\Delta})=7$, $\beta_2(A_{\Delta})=15$, $\beta_3(A_{\Delta})=13$ and $\beta_4(A_{\Delta})=4$ .The homological dimension of $A_{\Delta}$ is $hd_{A_{\Delta}}=4$ and the Krull dimension of $A_{\Delta}$ is $\dim(A_{\Delta})=7-hd_{A_{\Delta}}=3$ .Therefore, $A_{\Delta}$ is a Cohen-Macaulay of type $\beta_{hd_{A_{\Delta}}}=4$ .

**Definition 3.2 :**Given two $r$-arrangements $\mathrm{A}_1=\{H_1^1,\ldots,H_n^1\}$ and $\mathrm{A}_2=\{H_1^2,\ldots,H_n^2\}$ .

1 .We will say $\mathrm{A}_1$ and $\mathrm{A}_2$ have the same lattice or $L$-equivalent and denoted by $L(\mathrm{A}_1)\approx L(\mathrm{A}_2)$, if for each $1\le i_1<\cdots<i_k\le n$ and $1\le k\le n$ we have $rk(H_{i_1}^1,\ldots,H_{i_k}^1)=rk(H_{i_1}^2,\ldots,H_{i_k}^2)$ .

2 .For $2\le k\le r-1$, set $\Lambda_k(\mathrm{A}_i)=\{\mathrm{B}_i\subseteq \mathrm{A}_i \,\|\, B_i|\le k+1\}$ to be the lattice intersection pattern up to codimension $k$ of $\mathrm{A}_i$ and $i=1,2$ .We say $\mathrm{A}_1$ and $\mathrm{A}_2$ are $\Lambda_k$-equivalent and denoted by $\Lambda_k(\mathrm{A}_1)\approx\Lambda_k(\mathrm{A}_2)$ if for each $1\le i_1<\cdots<i_j\le n$ and $3\le j\le k+1$, we have $rk(H_{i_1}^1,\ldots,H_{i_j}^1)=rk(H_{i_1}^2,\ldots,H_{i_j}^2)$ .

3 .let $P_i(\mathrm{A}_i,t)$ be the Poincaré polynomial of $L(\mathrm{A}_i)$ and $i=1,2$ . $\mathrm{A}_1$ and $\mathrm{A}_2$ is said to be $P$-equivalent if $P_1(\mathrm{A}_1,t)=P_2(\mathrm{A}_2,t)$ .

Notice that, if $\mathrm{A}_1$ and $\mathrm{A}_2$ are $L$-equivalent, then they are $\Lambda_k$-equivalent for $2\le k\le r-1$ and they are $P$-equivalent .But the converse need not to be true in general. The following theorem of Ali, classified the hypersolvable class into a supersolvable subclass and the non supersolvable subclass by the minimal information that encoded in $\Lambda_2$:

**Theorem 3.2** ]1[ let $\mathrm{A}_1$ and $\mathrm{A}_2$ be supersolvable $r$-arrangements . $\mathrm{A}_1$ and $\mathrm{A}_2$ are $L$-equivalent if and only if they are $\Lambda_2$-equivalent .

**Theorem 3.3** If the supersolvable arrangements $\mathrm{A}_1$ and $\mathrm{A}_2$ are $\Lambda_2$-equivalent, then they have isomorphic matroids, i.e . $\mathrm{M}_{\mathrm{A}_1} = (\mathrm{A}_1, \Delta_1) \cong \mathrm{M}_{\mathrm{A}_2} = (\mathrm{A}_2, \Delta_2)$ .That is, they have isomorphic partition complexes, i.e . $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_1}) \cong BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_2})$ via the hypersolvable ordering which give rise into isomorphic standard $K$-algebras $A_{\Delta_1} \cong A_{\Delta_2}$ .

*Proof.* As a direct application of Ali's theorem )5.2(, $\mathrm{A}_1$ and $\mathrm{A}_2$ are $L$-equivalent .That is, there exists a one to one correspondence $\varphi : \mathrm{A}_1 \tilde{\to} \mathrm{A}_2$ satisfies for $1 \le k \le |A|$, $rk(H^1_{i_1}, \ldots, H^1_{i_k}) = c$ if, and only if, $rk(\varphi(H^1_{i_1}), \ldots, \varphi(H^1_{i_k})) = c$ .Thus, $\mathrm{A}_1$ and $\mathrm{A}_2$ have equivalent independent subarrangements .In particular, we define the isomorphism $\varphi : \Delta_1 \tilde{\to} \Delta_2$ between the $G$-complexes of $\mathrm{A}_1$ and $\mathrm{A}_2$ .Therefore, the maroids $\mathrm{M}_{\mathrm{A}_1} = (\mathrm{A}_1, \Delta_1)$ and $\mathrm{M}_{\mathrm{A}_2} = (\mathrm{A}_2, \Delta_2)$ are isomorphic.

On the other hand the one to one correspondence $\varphi : \mathrm{A}_1 \tilde{\to} \mathrm{A}_2$ which respects the HP analogue $\Pi_1$ on $\mathrm{A}_1$, define an equivalent HP $\Pi_2 = \varphi(\Pi_1)$ on $\mathrm{A}_2$ .In fact, $\varphi$ define equivalent hypersolvable orders on the hyperplanes of $\mathrm{A}_1$ and $\mathrm{A}_2$ which produce an isomorphism between the partition complexes $\varphi : BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_1}) \cong BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_2})$ via the hypersolvable ordering, since $\mathrm{A}_1$ and $\mathrm{A}_2$ have equivalent sets of sections .It's clear that the $G$-complexes of $\mathrm{A}_1$ and $\mathrm{A}_2$ have the same $f$-vectors and they have the same Hilbert functions which define an isomorphism between the graded $K$-algebras $A_{\Delta_{t_1}}$ and $A_{\Delta_{t_2}}$ .Where the equality $f^{\Delta_1} = f^{\Delta_2}$, define $K$-isomorphism between the free resolutions of $A_{\Delta_1}$ and $A_{\Delta_2}$ since they have the same Betti numbers .

In order to show that the fundamental group $\pi_1(M(\mathrm{A}))$ of the complement $M(\mathrm{A})$ has a fashion of an iterated almost direct product of free groups, Jambu and papadima defined a vertical deformation of a hypersolvable arrangement which is not fiber-type arrangement as follows:

**Theorem 3.4) :*Jambu's-Papadima's deformation Theorem*/2,3 (/** Let $\mathrm{A}$ be a hypersolvable $r$-arrangement such that $r < \ell$, i.e . $\mathrm{A}$ is not supersolvable .Then there is a vertical deformation $\{\tilde{\mathrm{A}}_t\}_{t \in C}$ of $\mathrm{A}$ in $C^r \times C^s = C^\ell$, where $s = \ell - r$ such that for each $t \in C$, $\tilde{\mathrm{A}}_t$ is supersolvable with $\mathrm{A}$ and $\tilde{\mathrm{A}}_t$ are $\Lambda_2$-equivalent.

Papadima and Suciu in ]13[, used Jambu's-Papadima's deformation $\{\tilde{\mathrm{A}}_t\}_{t \in C}$ of a hypersolvable arrangement $\mathrm{A}$ which is not supersolvable )not $K(\pi,1)$(, to show that the dimension of the first non vanishing higher homotopy group is;

$$p(M(\mathrm{A})) = sup\{k \mid P(H^*(M(\mathrm{A})),s) \equiv_{mod\, j} P(H^*(M(\tilde{\mathrm{A}}_1)),s), \forall j \le k\},$$

where $P(H^*(M(\mathrm{A})),s)$ and $P(H^*(M(\tilde{\mathrm{A}}_1)),s)$ are the Poincaré polynomials of the cohomological rings $H^*(M(\tilde{\mathrm{A}}_1))$ and $H^*(M(\tilde{\mathrm{A}}_1))$ respectively. Ali in ]1 [showed a conjecture of $p(M(\mathrm{A}))$ to produce a connection between the dimension of the first non vanishing higher homotopy group of a hypersolvable arrangement and the structure of it's lattice intersection with respect the hypersolvable ordering and the following definition represent her that :

**Definition 3.3 :**Let $\mathrm{A}$ be a hypersolvable $r$-arrangement with Hp $\Pi = (\Pi_1,...,\Pi_\ell)$ such that $r < \ell$, i.e . $\mathrm{A}$ is not supersolvable .Define $p(\mathrm{A}) = \max\{k \mid BC_{\trianglelefteq}^{k}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{k}(\mathrm{A})\}$.

Our purpose In this paper is to introduced a comparison between the structures of the broken circuit complexes of a hypersolvable arrangement and their Jambu's-Papadima's deformed fiber-type arrangements .

The principal significance of the next lemma is introducing the level $L_{p(\mathrm{A})+1}(\mathrm{A})$ as the first level in the intersection lattice $L(\mathrm{A})$ that contains a flat $X \in L_{p(\mathrm{A})+1}(\mathrm{A})$, such that the subarrangement $\mathrm{A}_X = \{H \in \mathrm{A} \mid \mathrm{X} \subseteq H\} \subseteq \mathrm{A}$ contains dependent sections among $(p(\mathrm{A})+2)$-different blocks of $\Pi$ via a hypersolvable order:

**Lemma 3.2** *Suppose we have the conclusions of definition )3.3 .(Then there exists* $S' \in S_{\Pi}^{p(\mathrm{A})+2}(\mathrm{A})$ *and* $S'$ *is a* $(p(\mathrm{A})+2)$ *-circuit.*

Proof :If $S \in S_{\Pi}^{p(\mathrm{A})+1}(\mathrm{A})$ such that $S \notin BC_{\trianglelefteq}^{p(\mathrm{A})+1}(\mathrm{M}_{\mathrm{A}})$, deduce that $S$ must be a $(p(\mathrm{A})+1)$-broken circuit .Let $H \in \mathrm{A}$ be the minimal hyperplane of $S$ via the hypersolvable ordering such that $S' = S \bigcup \{H\}$ forms a $(p(\mathrm{A})+2)$-circuit .Let $m = \min\{k \mid S \bigcap \Pi_k \neq \emptyset\}$ .It is clear if $S' \notin S_{\Pi}^{p(\mathrm{A})+2}(\mathrm{A})$, then $H \in \Pi_m$ .That applys the completeness property of $\Pi_m$ and produces a contradaction with our choice of $H$ .Therefore, $S' \in S_{\Pi}^{p(\mathrm{A})+2}(\mathrm{A})$ .

The following result is the main result of our paper which produces a comparison between the broken circuit complex of a hypersolvable arrangement which is not supersolvable and the broken circuit complexes of their deformed supersolvable arrangements:

**Theorem 3.5** *Let* $\mathrm{A}$ *be a hypersolvable* $r$ *-arrangement with Hp* $\Pi = (\Pi_1,...,\Pi_\ell)$, $d$ *-vector* $d = (d_1,...,d_\ell)$, $f$ *-vector* $f = (f_0, f_1,..., f_{r-1})$ *of* $\Delta$ *and* $f$ *-vector,* $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta},..., f_{\delta}^{\Delta})$ *of* $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ *such that* $r < \ell$ *.Then:*

1 *.For* $2 \le k \le r$, $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) \subseteq S_{\Pi}^{k}(\mathrm{A})$ *in general, i.e .* $f_{k-1}^{\Delta} \le \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \dots d_{i_k}$ .

2 *.* $BC_{\trianglelefteq}^{1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{2}(\mathrm{A})$, *i.e .* $f_1^{\Delta} = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}$.

*3 .There exists $2 \le p \le r-1$ such that $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{k}(\mathrm{A})$, for each and $L_{p+1}(\mathrm{A})$ represent the first level in the lattice intersection $L(\mathrm{A})$ that contains dependent relations among $p+2$-blocks of $\Pi$ .That is, for $2 \le k \le p$,*

$$f_{k-1}^{\Delta} = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k} .$$

*4 .For all $t_1, t_2 \in C \setminus \{0\}$, $\tilde{\mathrm{A}}_{t_1}$ and $\tilde{\mathrm{A}}_{t_2}$ are $L$-equivalent and they have isomorphic matroids $\mathrm{M}_{\tilde{\mathrm{A}}_{t_1}} = (\tilde{\mathrm{A}}_{t_1}, \tilde{\Delta}_{t_1})$ and $\mathrm{M}_{\tilde{\mathrm{A}}_{t_2}} = (\tilde{\mathrm{A}}_{t_2}, \tilde{\Delta}_{t_2})$ . That is, they have isomorphic partition complexes, i.e . $BC_{\trianglelefteq}(\mathrm{M}_{\tilde{\mathrm{A}}_{t_1}}) \cong BC_{\trianglelefteq}(\mathrm{M}_{\tilde{\mathrm{A}}_{t_2}})$ via the hypersolvable ordering which give rise into isomorphic standard $K$-algebras $A_{\Delta_{t_1}} \cong A_{\Delta_{t_2}}$ .*

*5 .For all $t \in C \setminus \{0\}$, $\mathrm{A}$ and $\tilde{\mathrm{A}}_t$ are $\Lambda_p$-equivalent, where*

$p = p(\mathrm{A}) = \max\{k \mid BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{k}(\mathrm{A})\}$.

*Thus Jambu's-Papadima's deformation preserves the lattice intersection pattern up to codimension $p$, then for $p(\mathrm{A})+1 \le k \le r$, the deformation destroyed all the flats $X \in L_k(\mathrm{A})$ that $\mathrm{A}_X$ contains dependent sections distrbuted among $j$-different blocks of $\Pi$, $j \ge k+1$, which adds new faces of $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ to deform it into the partition complex $S_{\tilde{\Pi}_t}(\tilde{\mathrm{A}}_t)$ as follows :*

*i. For $1 \le k \le p$, $\Delta_{k-1}$ and $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}})$ is invariant under the deformation .That is $\Delta_{k-1} \cong \Delta_{k-1}^{t}$ and $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) \cong S_{\tilde{\Pi}_t}^{k}(\tilde{\mathrm{A}}_t)$.*

*ii. For $p+1 \le k \le r$, Jambu's-Papadima's deformation replaced $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}})$ by $S_{\tilde{\Pi}_t}^{k}(\tilde{\mathrm{A}}_t)$ by adding exactly*

$$\{\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k}\} - f_{k-1}^{\Delta};$$

*$(k-1)$-faces .*

*iii. For $r+1 \le k \le \ell$, Jambu's-Papadima's deformation adding*

$$\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k};$$

*$(k-1)$-faces to obtain $S_{\tilde{\Pi}_t}^{k}(\tilde{\mathrm{A}}_t)$ .*

*That is, the G-complex and BC-complex of $\mathrm{A}$ embedded in the G-complex and partition complex of $\tilde{\mathrm{A}}_t$ respectively .Thus, for $0 \le k \le p+1$, $A_{\Delta}^{k} \cong A_{\tilde{\Delta}_t}^{k}$ and for $0 \le k \le p$, $M_k \cong M_k^t$ .*

*6 .If $r = 3$, then $p = 2$ and;*

$BC_{\trianglelefteq}^{2}(\mathrm{M}_{\mathrm{A}}) = \{\mathrm{B} \subseteq \mathrm{A} \mid \mathrm{B} \bigcap \Pi_1 \ne \varnothing \text{ and } |\mathrm{B}| = 3\}$,

*where;*

$f_2^{\Delta} = \sum_{i_1=2}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}.$

*Jambu's-Papadima's deformation keeps* $\Delta_0$, $BC_{\trianglelefteq}^{0}(\mathrm{M}_{\mathrm{A}})$, $\Delta_1$ *and* $BC_{\trianglelefteq}^{1}(\mathrm{M}_{\mathrm{A}})$ *unchanged and deform* $BC_{\trianglelefteq}^{2}(\mathrm{M}_{\mathrm{A}})$ *into* $S_{\tilde{\Pi}_t}^{3}(\tilde{\mathrm{A}}_t)$ *by adding exactly;*

$\sum_{i_1=2}^{\ell-2} \sum_{i_2=i_1+1}^{\ell-1} \sum_{i_3=i_2+1}^{\ell} d_{i_1} d_{i_2} d_{i_3},$

$2$ *-faces .Then, for* $4 \le k \le \ell$, *Jambu's-Papadima's deformation destroys all the dependent* $k$ *-sections of* $\mathrm{A}$ *and replaced it by;*

$\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k},$

$(k-1)$ *-faces to obtain* $S_{\tilde{\Pi}_t}^{k}(\tilde{\mathrm{A}}_t)$ *.That is for* $0 \le k \le 3$, $A_{\Delta}^{k} \cong A_{\tilde{\Delta}t}^{k}$ *and for* $0 \le k \le 2$, $M_k \cong M_k^t$ *where there is a monomorphism* $M_3 \hookrightarrow M_3^t$ *.*

*7 .If* $\mathrm{A}$ *is* $\ell$ *-generic, then* $p = r-1$ *and;*

$BC_{\trianglelefteq}^{r-1}(\mathrm{M}_{\mathrm{A}}) = \{\mathrm{B} \subseteq \mathrm{A} \mid \mathrm{B} \bigcap \Pi_1 \ne \varnothing \text{ and } |\mathrm{B}| = r\},$

*where* $f_{r-1} = \binom{\ell-1}{r-1}$. *Jambu's-Papadima's deformation letting* $\Delta_{k-1} = BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}})$ *invariant for* $1 \le k \le r-1$ *and added exactly* $\binom{\ell-1}{r}$, $r-1$ *-faces to deform* $BC_{\trianglelefteq}^{r-1}(\mathrm{M}_{\mathrm{A}})$ *into* $S_{\tilde{\Pi}_t}^{r}(\tilde{\mathrm{A}}_t)$ *.Then for* $r+1 \le k \le \ell$, *the deformation added* $\binom{\ell}{k}$, $k-1$ *-faces to deform* $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ *into* $S_{\tilde{\Pi}_t}$ *.That is for* $0 \le k \le r$, $A_{\Delta}^{k} \cong A_{\tilde{\Delta}t}^{k}$ *and for* $0 \le k \le r-1$, $M_k \cong M_k^t$ *where there is a monomorphism* $M_r \hookrightarrow M_r^t$ *.*

***Proof :***

**For 1 :**From] 1[, it was proved that "If $\mathrm{A}$ is a hypersolvable $r$-arrangement, then for $1 \le k \le r$, the $k$-no broken circuits of $\mathrm{A}$ must be distributed among $k$-different blocks", i.e .every $k$-no broken circuit of $\mathrm{A}$ is a $k$-section of $\Pi$ .That is the number of all $(k-1)^{th}$ faces of $\mathrm{A}$ can not exceed the number of all $k$-sections of $\Pi$ and our aim is hold .

**For 2 :**Every $2$-broken circuit is a $2$-section .On the other hand, if $\mathrm{B} = \{H_{i_1}, H_{i_2}\} \in S_{\Pi}^{2}(\mathrm{A})$ such that for $j = 1,2$, $H_{i_j} \in \Pi_{i_j}$ and $1 \le i_1 < i_2 \le \ell$, then from $i_1$-closeness property of $\Pi_{i_1}$, $\mathrm{B} \in BC_{\trianglelefteq}^{1}(\mathrm{M}_{\mathrm{A}})$ .Therefore, $BC_{\trianglelefteq}^{1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{2}(\mathrm{A})$ and $f_1^{\Delta} = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}$.

**For 3 :** By using Jambu's-Papadima's vertical deformation method, Ali in ]1 [showed that "for a hypersolvable $r$-arrangement which is not supersolvable, there exists $2 \le p \le r-1$ such that for each $1 \le k \le p$, every $k$-section of $\Pi$ forms a $k$-no broken circuit of $\mathrm{A}$ ", i.e. for $1 \le k \le p$, $BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{k}(\mathrm{A})$ and;

$f_{k-1}^{\Delta} = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k}$ .

**For 4 :**For all $t \in C \setminus \{0\}$, Jambu's-Papadima's vertical deformation method deformed the hypersolvable $r$-arrangement A into $\tilde{\mathrm{A}}_t$ which is a supersolvable $r$-arrangement with A and $\tilde{\mathrm{A}}_t$ are $\Lambda_2$-equivalent .That is for $t_1, t_2 \in C \setminus \{0\}$, $\tilde{\mathrm{A}}_{t_1}$ and $\tilde{\mathrm{A}}_{t_2}$ are $\Lambda_2$-equivalent and by applying theorem )3.2 (and theorem )3.3(, they are $L$-equivalent, they have isomorphic matroids, i.e . $\mathrm{M}_{\tilde{\mathrm{A}}_{t_1}} = (\tilde{\mathrm{A}}_{t_1}, \tilde{\Delta}_{t_1}) \cong \mathrm{M}_{\tilde{\mathrm{A}}_{t_2}} = (\tilde{\mathrm{A}}_{t_2}, \tilde{\Delta}_{t_2})$ and isomorphic partition complexes, i.e . $BC_{\trianglelefteq}(\mathrm{M}_{\tilde{\mathrm{A}}_{t_1}}) \cong BC_{\trianglelefteq}(\mathrm{M}_{\tilde{\mathrm{A}}_{t_2}})$ via equivalent hypersolvable orders which give rise into isomorphic standard $K$-algebras $A_{\Delta_{t_1}} \cong A_{\Delta_{t_2}}$ .

**For 5:** From )3.5.4 (above, the integer $p$ is defined to be $p = p(\mathrm{A}) = \max\{k \mid BC_{\trianglelefteq}^{k-1}(\mathrm{M}_{\mathrm{A}}) = S_{\Pi}^{k}(\mathrm{A})\}$ .Ali used this definition in ]1 [to showed that "for all $t \in C \setminus \{0\}$, A and $\tilde{\mathrm{A}}_t$ are $\Lambda_p$-equivalent ."Ali studied Jambu's-Papadima's vertical deformation $\{\tilde{\mathrm{A}}_t\}_{t \in C}$ of A by using the hypersolvable partition analogue and found that the deformation method destroys all the flats $X \in L_k(\mathrm{A})$, $2 < k \le r$ such that the subarrangement $\mathrm{A}_X = \{H \in \mathrm{A} \mid \mathrm{X} \subseteq H\} \subseteq \mathrm{A}$ contains dependent sections .Ali found a connection among the integer $p$, the hypersolvable partition and lattice intersection as; "the level $(p+1)$ in the lattice intersection $L(\mathrm{A})$ of A contains the first dependent relation among different blocks", i.e .there is a $(p+2)$-dependent section of $\Pi$ .Therefore, every $(p+1)$-blocks of the Hp $\Pi$ are independent . Then, "the deformation destroys all dependent relations of rank greater than or equal to $(p+1)$ among the blocks by destroying all the flats $X \in L_k(\mathrm{A})$, $p(\mathrm{A}) < k \le r$ such that the subarrangement $\mathrm{A}_X = \{H \in \mathrm{A} \mid \mathrm{X} \subseteq H\} \subseteq \mathrm{A}$ contains dependent sections ." That is the deformation method keeps all those sections of $\Pi$ which are no broken circuits invariant, where those sections of $\Pi$ which are not no broken circuits of A replaced by sections which are no broken circuits of $\mathrm{A}_t$ .In fact, the one to one correspondence $\varphi : \mathrm{A} \tilde{\to} \mathrm{A}_t$ which respects the lattice intersection pattern up to codimension two, $\varphi : \Lambda_2(\mathrm{A}) \tilde{\to} \Lambda_2 \tilde{\mathrm{A}}_t$, define a one to one correspondence $\varphi : \Pi \tilde{\to} \tilde{\Pi}_t$ with respect the same hypersolvable ordering of A and $\tilde{\mathrm{A}}_t$, )i.e . $\varphi$ gives each one of A and $\tilde{\mathrm{A}}_t$ the same $d$-vector .(Therefore,

$\varphi : S_{\Pi}(\mathrm{A}) \tilde{\to} S_{\tilde{\Pi}_t}(\tilde{\mathrm{A}}_t)$ ……)3.1(

forms a one to one correspondence, where it's restriction on $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$, $\varphi_{BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})} : BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}}) \tilde{\to} \varphi(BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}}))^{\circ}\ S_{\tilde{\Pi}_t}(\tilde{\mathrm{A}}_t)$ define an injection between the broken circuits complexes .That is, the deformation start with embedding $BC_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ as a subcomplex of $S_{\tilde{\Pi}_t}(\tilde{\mathrm{A}}_t)$, since all the no broken circuits of A are invariant under the deformation, then we can construct the broken circuit complex $S_{\tilde{\Pi}_t}(\tilde{\mathrm{A}}_t)$ of $\tilde{\mathrm{A}}_t$ by using the deformation method as follows :

**For )3.i :(**For $1 \le k \le p$, the equivalent lattice intersection pattern up to codimension $p$ of $\mathcal{A}$ and $\tilde{\mathcal{A}}_t$ give rise into a bijection $\varphi_{(k-1)} : \Delta_{(k-1)} \tilde{\to} \Delta^t_{(k-1)}$ of $(k-1)^{th}$-faces of the $G$-complexes since $\mathcal{A}$ and $\tilde{\mathcal{A}}_t$ have the same independent relations of rank $k$ and the restriction of $\varphi_{(k-1)}$ on $BC^{(k-1)}_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ is a bijection $\varphi_{(k-1)_{BC^{(k-1)}_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})}} : BC^{(k-1)}_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) \tilde{\to} S^k_{\Pi}(\mathcal{A})$ of broken circuit complexes which keep $BC^k_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ invariant under the deformation .Then, we added new faces to transform $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ into $S_{\tilde{\Pi}_t}(\tilde{\mathcal{A}}_t)$ and we can classified those added faces into two parts .

**For )3.ii :(**The first one is those faces which are the minimal nonfaces of $BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$ of dimension $(k-1)$, $p+1 \le k \le r$ which we add them by the restriction of )5.1 (on $S_{\Pi}(\mathcal{A}) \setminus BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})$,

$$\varphi_{S_{\Pi}(\mathcal{A})\setminus BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})} : S_{\Pi}(\mathcal{A}) \setminus BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}) \tilde{\to} \varphi(S_{\Pi}(\mathcal{A}) \setminus BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}}))^{\circ} \quad S_{\tilde{\Pi}_t}(\tilde{\mathcal{A}}_t),$$

i.e .we add exactly $|S_{\Pi}(\mathcal{A})| - |BC_{\trianglelefteq}(\mathrm{M}_{\mathcal{A}})|$, $(k-1)$-faces .

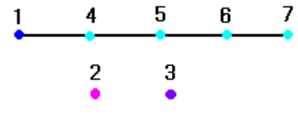
**For )3.iii :(**The second part contains all those faces of dimension $(k-1)$, $r+1 \le k \le \ell$, which we add them by the relation) 3.1( above, $\varphi_{(k-1)} : S^k_{\Pi}(\mathcal{A}) \tilde{\to} S^k_{\tilde{\Pi}_t}(\tilde{\mathcal{A}}_t)$, where the number of such faces is equal to $|S^k_{\Pi}(\mathcal{A})|$ .

Finally, for each $0 \le k \le p$, $\mathcal{A}$ and $\tilde{\mathcal{A}}_t$ have the same $f_k$, $f^{\Delta}_k$ and $H(\Delta, k+1)$ which is produced that for $0 \le k \le p+1$, $A^k_{\Delta} \cong A^k_{\tilde{\Delta}_t}$ and for $0 \le k \le p$, $M_k \cong M^t_k$ as free $A$-modules .

**For 6 :**If $r=3$, then each $3$-section $\mathcal{B}$ of $\Pi$ which not contains $H_1 \in \Pi_1$ is not $3$-no broken circuit of $\mathcal{A}$, since $\{H_1\} \bigcup \mathcal{B}$ is a $3$-circuit .Thus, $p=2$ and by applying )3.5.5 (above our aim is hold .

**For 7 :** If $\mathcal{A}$ is $\ell$-generic $r$-arrangement, then for each $1 \le k \le r-1$, every $k$-section $\mathcal{B}$ of $\Pi$ is a $k$-no broken circuit of $\mathcal{A}$ since for each $H \in \mathcal{A}$, the subarrangement $\{H\} \bigcup \mathcal{B}$ is independent .Where every $(r)$-section $\mathcal{X}$ of $\Pi$ which not contains $H_1 \in \Pi_1$ is not $r$-no broken circuit of $\mathcal{A}$, since $\{H_1\} \bigcup \mathcal{X}$ is a $r$-circuit .Thus, $p = r-1$ and by applying )3.5.5 (above, our claim is true . □

**Example 3.2 :**Let $\mathcal{A}$ be a hypersolvable $3$-arrangement defined by; $Q(\mathcal{A}) = z(y-x+z)(y+x+z)(y+3z)(y+2z)y(y-z)$. From the following cofigration of 7 points in the dual projective space $CP^{2*}$;




with the following hypersolvable ordering on $\mathcal{A}$; $H_1 = \ker(z)$, $H_2 = \ker(y-x+z)$, $H_3 = \ker(y+x+z)$, $H_4 = \ker(y+3z)$, $H_5 = \ker(y+2z)$, $H_6 = \ker(y)$ and $H_7 = \ker(y-z)$, the

HP of A be $\Pi=(\Pi_1,\Pi_2,\Pi_3,\Pi_4)=(\{H_1\},\{H_2\},\{H_3\},\{H_4,H_5,H_6,H_7\})$ which has a $d$-vector, $d=(1,1,1,4)$ .The matroid $M_{A}=(A,\Delta)$, is defined on A by letting $\Delta=\bigcup_{k=0}^{2}\Delta_k$, where;

$\Delta_0=\{1,\ldots,7\}$, $\Delta_1=\{\{i,j\}|1\le i<j\le 7\}$ and $\Delta_2=\{\{i,j,k\}|1\le i<j<k\le 7\}\setminus\{\{1,i,j\}|4\le i<j\le 7\}$.

That is the $f$-vector of $\Delta$ is $f=(7,21,25)$ and by simple calculations $H(\Delta,0)=1$, $H(\Delta,1)=7$, $H(\Delta,2)=28$, $H(\Delta,3)=74$, $H(\Delta,4)=145$ ...and the $h$-vector of $\Delta$ is $h=(1,4,10,8)$ since;

$$\sum_{m=0}^{\infty}H(\Delta,m)x^m=\frac{(1+4x+10x^2+8x^3)}{(1-x)^3}.$$

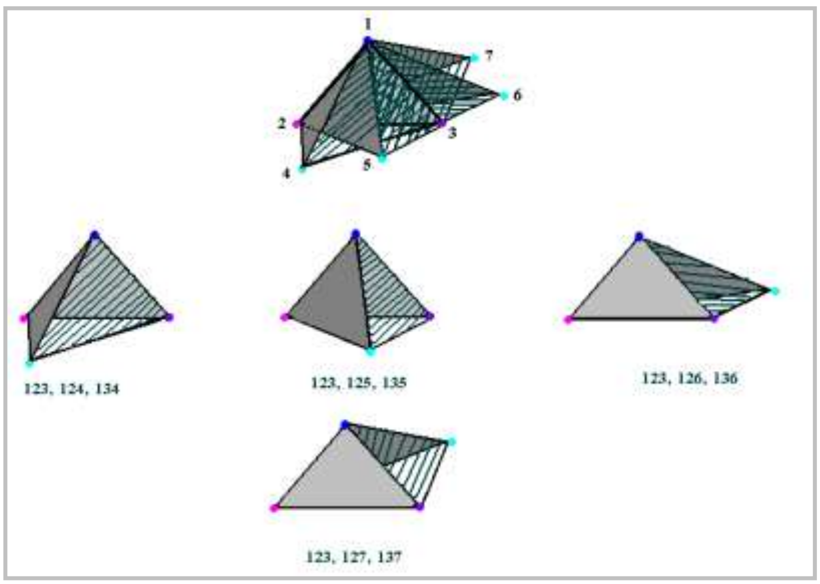
In fact A is not supersolvable and from theorem )3.1(,

$$BC_{\trianglelefteq}(M_{A})\neq S_{\Pi}(A).$$

By applying theorem )3.5(, the broken circuit complex $BC_{\trianglelefteq}(M_{A})$ of A, is defined as; $BC_{\trianglelefteq}(M_{A})=\bigcup_{k=0}^{2}BC_{\trianglelefteq}^{k}(M_{A})$, where;

$BC_{\trianglelefteq}^{0}(M_{A})=\{1,\ldots,7\}$, $BC_{\trianglelefteq}^{1}(M_{A})=\{\{i,j\}|1\le i<j\le 7\}\setminus\{\{i,j\}|4\le i<j\le 7\}$ and $BC_{\trianglelefteq}^{2}(M_{A})=\{\{1,j,k\}|2\le j<k\le 7\}\setminus\{\{1,j,k\}|4\le j<k\le 7\}$.

That is the $f$-vector of $BC_{\trianglelefteq}(M_{A})$ is $f^{\Delta}=(7,15,9)$ and figure )3.2.1 (includes a realization of $BC_{\trianglelefteq}(M_{A})$.

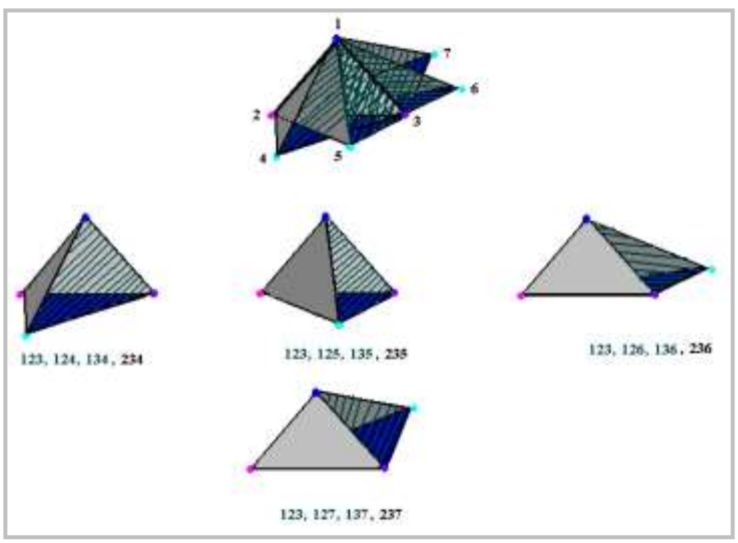



Figure(3.2.1)

Let $K$ be a field and let $A=K[H_1,\ldots,H_7]$ be the polynomial ring over $K$ whose variables are the vertices of $\Delta$ .Let $I_{\Delta}$ be the homogenous ideal of $A$ generated by the "minimal "non-faces of $\Delta$ .The standard $K$-algebra $A_\Delta = \sum_{m=0}^{\infty} A_\Delta^m$, of $\Delta$, as a graded algebra is determined by the Hilbert function of $A_\Delta$ as $H(A_\Delta, m) = \dim(A_\Delta^m) = H(\Delta, m)$ and the minimal finite free resolution of $A_\Delta$; $0 \to M_3 \to M_2 \to M_1 \to M_0 \to A_\Delta \to 0$,

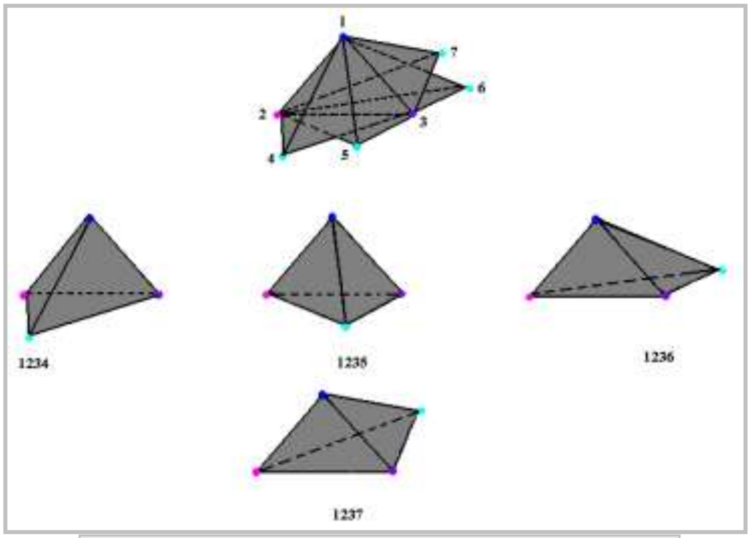
is completely determined by $f^\Delta$, since the Betti numbers of $A_\Delta$ are $\beta_0(A_\Delta) = 1$, $\beta_1(A_\Delta) = 7$, $\beta_2(A_\Delta) = 15$ and $\beta_3(A_\Delta) = 9$ .The homological dimension of $A_\Delta$ is $hd_{A_\Delta} = 3$ and the Krull dimension of $A_\Delta$ is $\dim(A_\Delta) = 7 - hd_{A_\Delta} = 4$ .Therefore, $A_\Delta$ is a Cohen-Macaulay of type $\beta_{hd_{A_\Delta}} = 9$.

By applying an algorithm given by Ali in ]1[, the supersolvable Jambu's-Papadima's vertical deformation $\{\widetilde{\mathrm{A}}_t\}_{t\in C}$ of $\mathrm{A}$ in $C^3 \times C = C^4$ is defined as $Q(\widetilde{\mathrm{A}}_t) = z(y-x+z)(y+x+z)(y+3z+tw)(y+2z+tw)(y+tw)(y-z+tw)$, for each $t \in C$ and by the same hypersolvable ordering of $\mathrm{A}$ let; $H_1^t = \ker(z)$, $H_2^t = \ker(y-x+z)$, $H_3^t = \ker(y+x+z)$, $H_4^t = \ker(y+3z+tw)$, $H_5^t = \ker(y+2z+tw)$, $H_6^t = \ker(y+tw)$ and $H_7^t = \ker(y-z+w)$, where the HP $\Pi_t = (\{H_1^t\},\{H_2^t\},\{H_3^t\},\{H_4^t, H_5^t, H_6^t, H_7^t\})$ of $\mathrm{A}_t$ have the same exponent vector with $\mathrm{A}$ .From Theorem )3.5.6(, $p = 2$ and Jambu's-Papadima's deformation keeps $BC^0_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$, $BC^1_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ unchanged, i.e .$BC^0_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_t}) = \{1^t, \ldots, 7^t\}$, $BC^1_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}_t}) = \{\{i^t, j^t\} \mid 1 \le i < j \le 7\} \setminus \{\{i^t, j^t\} \mid 4 \le i < j \le 7\}$ .Then we deform $BC^2_{\trianglelefteq}(\mathrm{M}_{\mathrm{A}})$ into $S^3_{\tilde{\Pi}_t}(\widetilde{\mathrm{A}}_t)$ by adding exactly four 2-faces, $\{2^t, 3^t, 4^t\}$, $\{2^t, 3^t, 5^t\}$, $\{2^t, 3^t, 6^t\}$ and $\{2^t, 3^t, 7^t\}$, as shown in figure(3.2.2) .




Figure(3.2.2)

Finally, Jambu's-Papadima's deformation destroys all the dependent $4$-sections of $\mathcal{A}$ and replaced it by four $3$-faces, $\{1^t,2^t,3^t,4^t\}$, $\{1^t,2^t,3^t,5^t\}$, $\{1^t,2^t,3^t,6^t\}$ and $\{1^t,2^t,3^t,7^t\}$ to obtain $S^4_{\tilde{\Pi}_t}(\tilde{\mathcal{A}}_t)$, as in figure(3.2.3) .




Figure(3.2.3)

Observe that if $t=1$, the supersolvable $4$-arrangement that given in example )3.1 (.forms the $1^{st}$-deformed arrangement $\tilde{\mathcal{A}}_1$ of $\mathcal{A}$ and from )3.5.4(, for all $t \in C\setminus\{0\}$, $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_t$ have the same lattice .That is we need only to compare $\mathcal{A}$ with $\tilde{\mathcal{A}}_1$ .Observe that, $\mathcal{A}$ and $\tilde{\mathcal{A}}_1$ have;

1 .the same lattice intersection pattern up to codimension two,
2 .the same HP,
3 .the same $d$-vector,
4 .for each $0 \le k \le 2$, $f_k^{\mathcal{A}} = f_k^{\mathcal{A}_1}$ and $f_k^{\Delta} = f_k^{\Delta_1}$ and
5 .for each $0 \le k \le 3$, $H(\Delta,k) = H(\Delta_1,k)$ .

By applying theorem )3.5.6(, we have the following commutative diagram of free $A$-modules :

$$\begin{array}{ccccccccccc} M_3 & \to & M_2 & \to & M_1 & \to & M_0 & \to & A_\Delta & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ M_3^t & \to & M_2^t & \to & M_1^t & \to & M_0^t & \to & A_{\tilde{\Delta}_t} & \to & 0; \end{array}$$

where for $0 \le k \le 2$, $M_k \cong M_k^t$ where there is a monomorphism $M_3 \hookrightarrow M_3^t$ which is defined a monomorphism $A_\Delta \hookrightarrow A_{\tilde{\Delta}_t}$ .

**Remark 3.1** For a hypersolvable arrangement $\mathcal{A}$, the quantities $\ell(\mathcal{A})$, the d-vector component "may in different order", the number of the resulting no broken circuits of $\mathcal{A}$ and the sections of the HP $\Pi$ are independent of our choice of the hypersolvable partition $\Pi$ and the order we used of the hyperplanes of $\mathcal{A}$ that caused by the geometric structure of the lattice intersection of $\mathcal{A}$ .The definition of Jambu's-Papadima's vertical deformation seems to be dependent of our choice of the related hypersolvable partition which needs not to be unique .But in the lattice intersection these deformed properties are dependent of the vertical deformation method that we used to deform A into a vertical family of supersolvable arrangements .

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# معقدة الدوارة المحطمة و معقدة التجزئة القابلة للحل فوقياً

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## المستخلص

تضمن هذ االبحث بناء معقدة الدوارة المحطمة لترتيبة $r$-قابلة للحل فوقياً A باستخدام التجزئة القابلة للحل فوقياً و الترتيب القابل للحل فوقياً الذي يحترم البنية القابلة للحل فوقياً. وباستخدام فقط الحد الادنى من المعلومات الممثلة بشبكة التقاطعات الى المستوى اثنين، تمكنا من اعطاء تكافؤ بين معقدات الدوارات المحطمة لترتيبات قابلة للحل كلياً و الذي مكننا من اعطاء مقارنة بين البنى لمعقدات الدوارات المحطمة لكل من الترتيبة قابلة للحل فوقياً و ترتيبات جامبو و باباديما المشوهة ذات النوع الليفي العائدة لها.